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# SYMBOLIC ESTIMATION OF DISTANCES BETWEEN EIGENVALUES OF A HERMITIAN OPERATOR

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**Abstract**. We find a method for symbolic estimation of a minimal (maximal) distance between (different) eigenvalues of a Hermitian operator (of a Hermitian matrix) using Hankel matrices formalism.

**Keywords and phrases**: Hermitian matrix, characteristic polynomial, Hankel matrices. Multiplicity of eigenvalue.

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**1 Introduction.** Localization of roots of polynomial was considered by a number of authors: C. Jacobi, K. Mahler, S. Gershgorin, Yu. Uteshev, etc. In contrast to these authors we restricted ourselves with polynomials having real roots only, but we have considered degenerate roots too. The main method we have used was proposed in 1996 [1]. It allows us to define the number of different roots and to calculate the degree of degeneracy (the multiplicity) of the certain root. The constructed algorithm allows us to construct the recursive sequences for minimal and maximal distances between roots. The limits of these sequences are exact values of minimal and maximal distances. Each term of the sequences is rational function of the coefficients of the polynomial.

**2 Main result.** Considering a polynomial $P_n(x)$ with real roots $p_1 < p_2 < ... < p_m$ having multiplicities $r_1, r_2, ..., r_m$, $\sum\limits_{i=1}^{m} r_i = n$, we derive (see [1]) the relation:

$$r_i{}^{-1} = D_m{}^{-1} \det \begin{bmatrix} H_m & (\mathbf{p}_i)^T \\ -\mathbf{p}_i & 0 \end{bmatrix}, \tag{1}$$

where

$$H_k = [h_{ij}]_1^k = [t_{i+j-2}]_1^k, \ \ t_k = \sum_{l=1}^{m} r_l p_l^k, \ \ k = 0, 1, 2, ..., \tag{2}$$

and $\mathbf{p}_i \equiv \left[p_i^{j-1}\right]_{j=\overline{1,m}}$ denotes one-row matrix, $\mathbf{p}_i^T \equiv \left[p_i^{j-1}\right]_{j=\overline{1,m}}^T$ - correspondent one-column matrix. One has (see [1]):

$$D_k = \det H_k = \begin{cases} \sum\limits_{1 \le i_1 < ... < i_k \le m} \left\{ r_{i_1} \cdots r_{i_k} \prod\limits_{1 \le j < l \le k} (p_{i_j} - p_{i_l})^2 \right\} > 0, & 1 \le k \le m, \\ 0, & k > m. \end{cases} \tag{3}$$

$$D_2 = \det\left(\begin{bmatrix} 1 & \dots & 1 \\ p_1 & \dots & p_m \end{bmatrix}\begin{bmatrix} r_1 & \dots & r_m \\ r_1 p_1 & \dots & r_m p_m \end{bmatrix}^T\right) = \sum_{1\le i<j\le m} r_i r_j (p_j - p_i)^2. \qquad (4)$$

Calculating the determinant in (1), we obtain

$$r_i^{-1} = \left\langle \mathbf{p}_i {H_m}^{-1} \mathbf{p}_i^T \right\rangle, \ \ r_i^{-1}\delta_{ij} = \left\langle \mathbf{p}_i {H_m}^{-1} \mathbf{p}_j^T \right\rangle, \qquad (5)$$

where brackets stand for a scalar product of a row by a column and $\delta_{ij}$ denotes the Kronecker delta. It is easy to show, that

$$r_j^{-1} - r_i^{-1} = \left\langle (\mathbf{p}_j + \mathbf{p}_i) {H_m}^{-1} (\mathbf{p}_j - \mathbf{p}_i)^T \right\rangle. \qquad (6)$$

Expanding the difference of degrees in the last column in the formula (6) we obtain

$$p_j^{k-1} - p_i^{k-1} = (p_j - p_i) \sum_{l=0}^{k-2} p_i^{k-2-l} p_j^l, \quad k = \overline{2,m}. \qquad (7)$$

Let us construct the expression

$$Z \equiv \sum_{i<j} \frac{r_j^{-1} - r_i^{-1}}{p_j - p_i} = \sum_{i<j} \left\{ \sum_{n,k=1}^{m} ({p_j}^{n-1} + {p_i}^{n-1}) [{H_m}^{-1}]_{nk} \sum_{l=0}^{k-2} p_i^{k-2-l} p_j^l \right\}. \qquad (8)$$

Obviously, the right hand side of it is a symmetric function of $p_k$. So, it can be expressed as a rational function of the coefficients of the polynomial $P_m(x) = \prod_{i=1}^{m} (x - p_i)$. It is easy to check that this polynomial can be expressed in terms of coefficients of the polynomial $P_n(x)$ as:

$$P_m(x) = D_m^{-1} \det \begin{bmatrix} t_0 & t_1 & \dots & t_{m-1} & t_m \\ t_1 & t_2 & \dots & t_m & t_{m+1} \\ \dots & \dots & \dots & \dots & \dots \\ t_{m-1} & t_m & \dots & t_{2m-2} & t_{2m-1} \\ 1 & x & \dots & x^{m-1} & x^m \end{bmatrix}. \qquad (9)$$

Applying formula (8) to the polynomial

$$Q_{3m}(x) = P_m(x) P_m^2(x - \varepsilon) = \prod_{i=1}^{m} (x - p_i) \prod_{j=1}^{m} (x - (p_j + \varepsilon))^2, \qquad (10)$$

and assuming that $\varepsilon$ is chosen according to the condition $p_i - (p_j + \varepsilon) \ne 0, \ i, j = \overline{1,m}$, after obvious transformations one gets

$$Z(\varepsilon) = \frac{1}{2}\left( -\frac{m}{\varepsilon} + \sum_{1\le i<j\le m} \frac{2\varepsilon}{(p_i - p_j)^2 - \varepsilon^2} \right). \qquad (11)$$

Let us assume that $\varepsilon = \mu_0,\ \ 0 < \mu_0 < \min|p_i - p_j| \equiv \mu$. Namely, we can choose

$$\mu_0{}^2 = \left[\sum_{1\le i<j\le m} (p_i - p_j)^{-2}\right]^{-1} < \min{(p_i - p_j)^2}. \tag{12}$$

Then one obtains

$$\frac{1}{\mu_0}\left(Z(\mu_0) + \frac{m}{2\mu_0}\right) = \sum_{1\le i<j\le m} \frac{1}{(p_i - p_j)^2 - \mu_0{}^2} > \frac{1}{\mu^2 - \mu_0{}^2} > 0. \tag{13}$$

As a result, the next estimation follows from (13)

$$\mu^2 > \left(\sum_{1\le i<j\le m} \frac{1}{(p_i - p_j)^2 - \mu_0{}^2}\right)^{-1} + \mu_0{}^2 \equiv \mu_1{}^2. \tag{14}$$

Obviously, $\mu_0{}^2 < \mu_1{}^2 < \mu^2$. Continuing this process, after $k$ steps one obtains the increasing and above bounded sequence

$$\mu_0{}^2 < \mu_1{}^2 < \mu_2{}^2 < ... < \mu_k{}^2 < \mu^2, \tag{15}$$

$$\mu_k{}^2 \equiv \left(\sum_{1\le i<j\le m} \frac{1}{(p_i - p_j)^2 - \mu_{k-1}{}^2}\right)^{-1} + \mu_{k-1}{}^2, \quad k = 1, 2, ... \,. \tag{16}$$

When $k \to \infty$ the sequence (15) must converge to some limit $\lim\limits_{k\to\infty} \mu_k = \mu_\infty, 0 < \mu_\infty \le \mu$. This limit can be easily calculated using the recurrent relation (16): assuming $k \to \infty$ in the both sides of it, one obtains

$$\mu_\infty{}^2 \equiv \left(\sum_{1\le i<j\le m} \frac{1}{(p_i - p_j)^2 - \mu_\infty{}^2}\right)^{-1} + \mu_\infty{}^2. \tag{17}$$

Hence, $\sum\limits_{1\le i<j\le m} \frac{1}{(p_i-p_j)^2-\mu_\infty{}^2} \to \infty$. So, in the sum at least one summand is infinite i. e. at least one denominator in it is equal to 0. Taking into account that $0 < \mu_\infty \le \mu \equiv \min|p_i - p_j|$, one has to conclude $\mu_\infty = \mu = \min|p_i - p_j|$.

Similarly, assuming in the formula (11) $\varepsilon = M_0 > \max|p_i - p_j| = p_m - p_1 \equiv M > 0$, one gets bellow bounded sequence:

$$M^2 < M_k{}^2 < ... < M_2{}^2 < M_1{}^2 < M_0{}^2, \tag{18}$$

$$M^2 < \left(\sum_{1\le i<j\le m} \frac{1}{(p_i - p_j)^2 - M_{k-1}{}^2}\right)^{-1} + M_{k-1}{}^2 \equiv M_k{}^2. \tag{19}$$

For $M_0 > \max|p_i - p_j|$ one can choose $M_0{}^2 = D_2$ (see formula (4)).

The sequence (15) must converge to some limit $\lim_{k\to\infty} M_k = M_\infty,\ 0 < M_\infty \le M$, when $k \to \infty$. Assuming $k \to \infty$ in the both sides of it, one obtains $M_\infty = M = \max|p_i - p_j|$.

It is obvious that the sum in formula (12) is a symmetric function of the roots $p_1, p_2, .., p_m$ of the polynomial $P_m(x)$. Therefore, $\mu_0^2$, as well as $M_0^2$, can be expressed as a rational function of the coefficients of the polynomial $P_n(x)$. So, we have proven the following

**Theorem.** *The minimal (the maximal) distance between the roots of the polynomial (with real roots) can be found as the limit of an increasing (decreasing) convergent sequence of the rational functions of the coefficients of the polynomial only.*

## 3 Conclusions.

As far as a characteristic polynomial of a Hermitian matrix has real roots only, it is shown that maximal and minimal distances between eigenvalues of a Hermitian matrix (which are roots of the polynomial) can be expressed as limits of convergent sequences, each term of which can be expressed rationally by the characteristic polynomial's coefficients - unitary invariants of the Hermitian matrix.

## REFERENCES

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