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ELABORATION OF THE ARNOLD THEOREM ON ELLIPSOIDS' SPACE CO-DIMENSION

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Abstract. We have shown that a set of revolution ellipsoids forms a boundary of the space of generic ellipsoids and therefore does not divide the generic ellipsoids' space (in agreement with Arnold). Nevertheless, it is shown that in order to fix a position of a point on this boundary - an ellipsoid of revolution - an additional discrete parameter is necessary (in contradiction with the Arnold Theorem).

Keywords and phrases: Ellipsoid, Arnold theorem, co-dimension.

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1 Introduction. Arnold theorem about the co-dimensions of revolutionellipsoids subset in the space of generic ellipsoids is considered. While proving the theorem Arnold shows that $(n^2 + n - 4)/2$ parameters are needed to define the revolution ellipsoid with two times degenerated one axis. In our opinion, one more discrete parameter is required. In spite of this the corollary of this theorem which states that revolution ellipsoids' subset doesn't divide the generic ellipsoids' space (as well as the line doesn't divide the three-dimensional space) is valid.

2 Main result. The following theorem takes place.

Theorem. *The set of revolution ellipsoids is a finite union of smooth submanifolds of co-dimension not less than 2 in the space of all ellipsoids [1].*

The number of parameters which define non degenerated ellipsoid in $\mathbb{R}^n$ is $n\,(n+1)/2$ (number of coefficients of a quadratic form). While proving the Theorem V. Arnold writes: if one of the axis is two times degenerated then we need lengths of the axes:

$$0 < p_1 < p_2 < \ldots < p_{n-1} \tag{1}$$

and

$$(n-1) + (n-2) + ... + 2 = \frac{(n+1)(n-2)}{2} \tag{2}$$

angles to fix the orientation of the ellipsoid. So it needs $(n^2 + n - 4)/2$ parameters only and co-dimension of this space in the space of generic ellipsoids is 2. Our statement is that we need one more discrete parameter: one has to specify the axis p_i which is degenerated.

Example. Without this parameter we cannot distinguish between the two (3 dimensional) ellipsoids of revolution shown in Fig. 1. In spite of this we agree with the Corollary

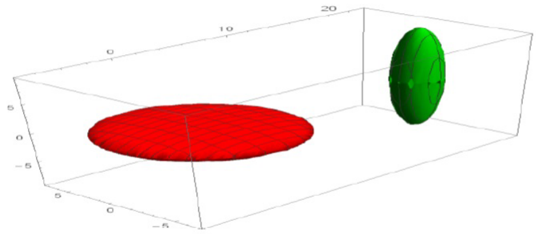

Figure 1:

of the Arnold Theorem except last words (the correspondent part of the Corollary in the Arnold's formulation below is bolded)

Corollary. (Arnold): Let us consider the space of ellipsoids (in $\mathbb{R}^n$) having dimension $n\,(n+1)/2$, each point of which represents a generic ellipsoid. Any two points of the space can be connected by a continuous line in this space which does not pass the points representing degenerated ellipsoids; **this happens due to the fact that the space of degenerated ellipsoids is a finite union of smooth submanifolds of co-dimension not less than 2 in the space of all ellipsoids.** In contrast to V. Arnold, to prove the corollary we have considered the canonical form of ellipsoids, in agreement with Klein's Erlangen Program that claims that any geometrical object is invariant of some group; and geometrical theorems are relations between the invariants of the group (Euclidean geometry corresponds to the group of orthogonal transformations). So we have to characterize an ellipsoid by its orthogonal invariants only, especially, by the lengths of its axes.

Example. Let us consider a canonical form of an ellipsoid in $\mathbb{R}^3$. It is defined only by the lengths ellipsoid's axes: $p_1 \le p_2 \le p_3$.

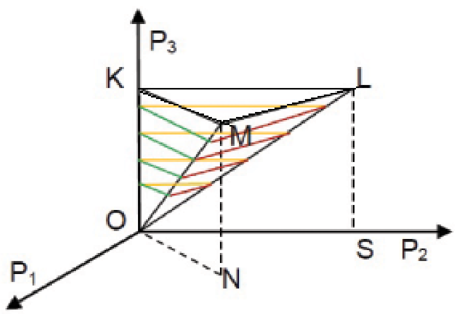

Figure 2:

Degenerated ellipsoids form the boundary of the space of all ellipsoids (Fig. 2), hence it cannot divide the space of ellipsoids. This is in agreement with the Corollary of the Arnold Theorem. The similar situation we have for ellipsoids in $\mathbb{R}^n$, $n > 3$.

3 Conclusions. While proving the theorem V. Arnold says that to define the ellipsoid of revolution one needs the lengths of its axis and values of angles (to fix the ellipsoid's orientation in the Euclidian space). We have shown that it is necessary to specify namely which axes are degenerated. We considered the canonical form of ellipsoids and have shown that in this case the subset of ellipsoids of revolution form the boundary of the space of all ellipsoids' space, hence this subset cannot divide the space of ellipsoids (in agreement with the corollary of the Arnold theorem).

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