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ESTIMATION OF THE EQUILIBRIUM AND OF THE DEGREE OF VOLATILITY OF THE ORNSTEIN-UHLENBECK'S STOCHASTIC PROCESS *

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Abstract. By using the Kolmogorov's strong law of large numbers, the consistent estimates of the equilibrium and of the degree of volatility are constructed in the Ornstein-Uhlenbeck's stochastic model.

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1 Introduction.

The Ornstein-Uhlenbeck process, x_t satisfies the following stochastic differential equation:

$$dx_t = \theta(\mu - x_t)dt + \sigma dW_t, \tag{1}$$

where $\theta > 0$, $\mu \in R$ and $\sigma > 0$ are parameters and W_t denotes the Wiener process.

The solution of the stochastic differential equation (1) has the following form

$$x_t = x_0 e^{-\theta t} + \mu(1 - e^{-\theta t}) + \sigma \int_0^t e^{-\theta(t-s)} dW_s, \tag{2}$$

where x_0 is assumed to be constant.

The parameters in (2) have the following sense:

(i) μ represents the equilibrium or mean value supported by fundamentals (in other words, the central location);

(ii) σ is the degree of volatility around it caused by shocks;

(iii) θ is the rate by which these shocks dissipate and the variable reverts towards the mean;

(iv) x_0 is the underlying asset price at moment $t = 0$ (the underlying asset initial price);

(v) x_t is the underlying asset price at moment $t > 0$.

There are various scientific papers devoted to estimate of parameter μ, σ and θ(see, for example [1], [2]). There least-square minimization and maximum likelihood estimation techniques are used for the estimating parameters σ and μ which work successfully. The same we can not say concerning the estimating the parameter θ (see, for example, [1]).

The purpose of the present paper is to introduce a new approach which by use of values $(z_k)_{k\in N}$ of corresponding trajectories at a fixed positive moment t, will allows us

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*This article is dedecated to the memory of the Tbilisi State University Professor Grigol Sokhadze to construct a consistent estimate of parameters σ and μ of the Ornstein-Uhlenbeck's stochastic process under an assumption that all another parameters are known.

The rest of the present paper is organized as follows.

In Section 2 we consider some auxiliary notions and facts from the theory of stochastic differential equations and mathematical statistics.

In Section 3 we present the constructions of consistent estimates for unknown parameters σ and μ in the Ornstein-Uhlenbeck's stochastic model.

2 Some auxiliary facts from the theory of stochastic differential equations and mathematical statistics. By use of approaches introduced in [3] one can get the validity of the following Lemmas

Lemma 1. *Let's consider an Ornstein-Uhlenbeck process x_t satisfies the following stochastic differential equation:*

$$dx_t = \theta(\mu - x_t)dt + \sigma dW_t \tag{3}$$

where $\theta > 0$, μ and $\sigma > 0$ are parameters and W_t denotes the Wiener process. Then the solution of this stochastic differential equation (3) is given by

$$x_t = x_0 e^{-\theta t} + \mu(1 - e^{-\theta t}) + \sigma \int_0^t e^{-\theta(t-s)} dW_s,$$

where x_0 is assumed to be constant.

Lemma 2. *Under conditions of Lemma 1, the following equalities*

(i) $E(x_t) = x_0 e^{-\theta t} + \mu(1 - e^{-\theta t})$;

(ii) $cov(x_s, x_t) = \frac{\sigma^2}{2\theta}\left(e^{-\theta(t-s)} - e^{-\theta(t+s)}\right)$;

(iii) $var(x_s) = \frac{\sigma^2}{2\theta}\left(1 - e^{-2\theta s)}\right)$;

hold true.

Lemma 3. *(Kolmogorov's strong law of large numbers [4]) Let $X_1, X_2, ...$ be a sequence of independent identically distributed random variables defined on the probability space $(\Omega, \mathcal{F}, P)$. If these random variables have a finite expectation m (i.e., $E(X_1) = E(X_2) = ... = m < \infty$), then the following condition*

$$P(\{\omega : \lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} X_k(\omega) = m\}) = 1$$

holds.

3 Main results. We begin this section by the following definition.

Definition. A Borel measurable function $T_n : R^n \to R$ $(n \in N)$ is called a consistent estimator of a parameter θ (in the sense of almost everywhere convergence) for the family $(\mu_\theta^N)_{\theta \in R}$ if the following condition

$$\mu_\theta^N(\{(x_k)_{k\in N} : \ (x_k)_{k\in N} \in R^N \ \& \ \lim_{n\to\infty} T_n(x_1, \cdots, x_n) = \theta\}) = 1$$

holds true for each $\theta \in R$.

By the use of Kolmogorov's Strong Law of Large numbers the validity of the following assertion is obtained.

Theorem 1. *For $t > 0$, $x_0 \in R$, $\theta > 0$, $\mu \in R$ and $\sigma > 0$, let's $\gamma_{(t,x_0.\theta,\mu,\sigma)}$ be a Gaussian probability measure in R with the mean $m_t = x_0 e^{-\theta t} + \mu(1 - e^{-\theta t})$ and the variance $\sigma_t^2 = \frac{\sigma^2}{2\theta}\left(1 - e^{-2\theta s}\right)$. Assuming that parameters x_0, t, θ and σ are fixed, for $\mu \in R$ let's denote by γ_μ the measure $\gamma_{(t,x_0.\theta,\mu,\sigma)}$. Let us define the estimate $T_n^* : R^n \to R$ by the following formula*

$$T_n^*((z_k)_{1 \le k \le n}) = \left(\frac{\sum_{k=1}^n z_k}{n} - x_0 e^{-\theta t}\right) / \left(1 - e^{-\theta t}\right).$$

Then we get

$$\gamma_\mu^\infty\{(z_k)_{k \in N} : (z_k)_{k \in N} \in R^\infty \ \& \ \lim_{N \to \infty} T_n((z_k)_{1 \le k \le n}) = x_0\} = 1,$$

provided that T_n is a consistent estimator of the equilibrium $\mu \in R$ in the sense of almost everywhere convergence for the family of probability measures $(\gamma_\mu^\infty)_{\mu \in R}$.

Proof. Let's consider probability space $(\Omega, \mathcal{F}, P)$, where $\Omega = R^\infty$, $\mathcal{F} = B(R^\infty)$, $P = \gamma_\mu^\infty$.

For $k \in N$ we consider k-th projection Pr_k defined on R^∞ by

$$Pr_k((x_i)_{i \in N}) = x_k$$

for $(x_i)_{i \in N} \in R^\infty$.

It is obvious that $(Pr_k)_{k \in N}$ is a sequence of independent Gaussian random variables with the mean $m_t = x_0 e^{-\theta t} + \mu(1 - e^{-\theta t})$ and the variance $\sigma_t^2 = \frac{\sigma^2}{2\theta}\left(1 - e^{-2\theta s}\right)$. By the use of Kolmogorov's Strong Law of Large numbers we get

$$\gamma_\mu^\infty\{(z_k)_{k \in N} \in R^\infty \ \& \ \lim_{n \to \infty} \frac{\sum_{k=1}^n Pr_k((z_k)_{k \in N})}{n} = x_0 e^{-\theta t} + \mu(1 - e^{-\theta t})\} = 1,$$

which implies

$$\gamma_\mu^\infty\{(z_k)_{k \in N} \in R^\infty \ \& \ \lim_{n \to \infty} \left(\frac{\sum_{k=1}^n z_k}{n} - x_0 e^{-\theta t}\right) / \left(1 - e^{-\theta t}\right) = \mu\}$$

$$= \gamma_\mu^\infty\{(z_k)_{k \in N} \in R^\infty \ \& \ \lim_{n \to \infty} T_n^*((z_k)_{1 \le k \le n}) = \mu\} = 1.$$

By the scheme used in the proof of Theorem 1, one can get the validity of the following theorem.

Theorem 2. *For $t > 0$, $x_0 \in R$, $\theta > 0$, $\mu \in R$ and $\sigma > 0$, let $\gamma_{(t,x_0.\theta,\mu,\sigma)}$ be a Gaussian probability measure in R with the mean $m_t = x_0 e^{-\theta t} + \mu(1 - e^{-\theta t})$ and the variance $\sigma_t^2 = \frac{\sigma^2}{2\theta}\left(1 - e^{-2\theta s}\right)$. Assuming that parameters x_0, t, μ and θ are fixed. For $\sigma^2 > 0$, let's denote by γ_{σ^2} the measure $\gamma_{(t,x_0,\theta,\mu,\sigma)}$. Let us define the estimate $T_n^{***} : R^n \to R$ by the following formula*

$$T_n^{***}((z_k)_{1 \le k \le n}) = \frac{2\theta \sum_{k=1}^n \left(z_k - x_0 e^{-\theta t} - \mu(1 - e^{-\theta t})\right)^2}{n\left(1 - e^{-2\theta s}\right)}.$$

Then we get

$$\gamma_{\sigma^2}^{\infty}\{(z_k)_{k \in N} : (z_k)_{k \in N} \in R^{\infty} \ \& \ \lim_{N \to \infty} T_n^{***}((z_k)_{1 \le k \le n}) = \sigma^2\} = 1,$$

provided that T_n^{***} is a consistent estimator of the square of the degree of volatility σ around it caused by shocks in the sense of almost everywhere convergence for the family of probability measures $(\gamma_{\sigma^2}^{\infty})_{\sigma^2 > 0}$.

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