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FINITE DIFFERENCE SCHEME FOR ONE NONLINEAR PARTIAL INTEGRO-DIFFERENTIAL EQUATION *

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Abstract. The paper concerns the investigation of finite difference scheme for nonlinear partial integro-differential equation which is based on system of Maxwell equations describing the process of propagation of the electromagnetic field into a substance. A wider class of nonlinearity is studied than the one that has been investigated before.

Keywords and phrases: Nonlinear partial integro-differential equation, initial-boundary value problem, asymptotic behavior, finite difference scheme, convergence.

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In the domain $[0,1]\times[0,\infty)$ let us consider the following initial-boundary value problem:

$$\frac{\partial U}{\partial t}=\frac{\partial}{\partial x}\left[a\left(\int\limits_0^t\left(\frac{\partial U}{\partial x}\right)^2d\tau\right)\frac{\partial U}{\partial x}\right],\tag{1}$$

$$U(0,t)=U(1,t)=0,\tag{2}$$

$$U(x,0)=U_0(x),\tag{3}$$

where U_0 is a given function and $a=a(S)$ is defined for $S\in[0;\infty)$.

Integro-differential equations of parabolic type arise in the study of various problems (see, for example, [1], [7], [13], [14], [20] and references therein). One such model is obtained at mathematical modeling of processes of electromagnetic field penetration in the substance. It is shown that in quasi-stationary approximation the corresponding system of Maxwell equations [15] can be rewritten in the following form [6]:

$$\frac{\partial H}{\partial t}=-rot\left[a\left(\int\limits_0^t|rotH|^2\,d\tau\right)rotH\right],\tag{4}$$

where $H=(H_1,H_2,H_3)$ is a vector of the magnetic field.

Note that integro-differential models of (4) type are complex and still yield to the investigation only for special cases (see, for example, [2]-[6], [8]-[13], [16], [17], [19], [21] and references therein).

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Study of the models of type (4) have begun in [6]. In particular, for the case $a(S) = 1 + S$ the theorems of existence of solution of the first boundary value problem for scalar and one-dimensional space case and uniqueness for more general cases are proved in that work. One-dimensional scalar variant for the case $a(S) = (1 + S)^p$, $0 < p \leq 1$ is studied in [3]. Investigations for multi-dimensional space cases at first was carried out in [4]. Multidimensional space cases are also discussed in [10], [17].

Asymptotic behavior as $t \to \infty$ of solutions of initial-boundary value problems for (4) type models are studied in [5], [10]-[13] and in a number of other works as well. In those works main attentions are paid to one-dimensional analogs.

Finite element analogs and Galerkin method algorithm as well as settling of semi-discrete and finite difference schemes for (4) type one-dimensional integro-differential models are studied in [8], [12], [18], [21], [22] and in the other works as well (see [13] and references therein).

If the magnetic field has the form $H = (0, 0, U)$, $U = U(x, t)$, then from (4) we obtain the integro-differential equation (1) studied in this note.

Our main aim is to study the finite difference scheme of initial-boundary value problem (1)-(3). Attention is paid to the investigation of wider cases of nonlinearity than already were studied. In particular, we consider the case when $a(S) = (1 + S)^p$, $0 < p \leq 1$. The theorem of asymptotic behavior of solution is stated as well.

Using the compactness method, a modified version of the Galerkin method [20], [24] the unique solvability can be proved.

Let us note that same results are true for the problem with first type homogeneous conditions on the whole boundary (see, for example, [10], [13] and references therein).

The following theorem of asymptotic stability of solution takes place.

Theorem 1. *If $a(S) = (1 + S)^p$, $0 < p \leq 1$ and $U_0 \in H^3(0, 1)$, $U_0(0) = U_0(1) = 0$, then for the solution of problem (1) - (3) the following estimates hold as $t \to \infty$:*

$$\left| \frac{\partial U(x,t)}{\partial x} \right| \leq C \exp\left(-\frac{t}{2}\right), \qquad \left| \frac{\partial U(x,t)}{\partial t} \right| \leq C \exp\left(-\frac{t}{2}\right),$$

uniformly in x on $[0, 1]$.

Now let us consider the finite difference scheme for problem (1) - (3) for the case $a(S) = (1+S)^p$, $0 < p \leq 1$. On $[0, 1]\times[0, T]$, where T is a positive number, let us introduce a net with mesh points denoted by $(x_i, t_j) = (ih, j\tau)$, where $i = 0, 1, ..., M;\ j = 0, 1, ..., N$ with $h = 1/M$, $\tau = T/N$. The initial line is denoted by $j = 0$. The discrete approximation at (x_i, t_j) is designed by u_i^j and the exact solution to problem (1) - (3) by U_i^j. We will use the following known notations [23]:

$$u_{x,i}^j = \frac{u_{i+1}^j - u_i^j}{h}, \quad u_{\bar{x},i}^j = \frac{u_i^j - u_{i-1}^j}{h}, \quad u_{t,i}^j = \frac{u_i^{j+1} - u_i^j}{\tau}.$$

Introduce the inner product and the norm:

$$(u^j, v^j) = h \sum_{i=1}^{M-1} u_i^j v_i^j, \quad \|u^j\| = (u^j, u^j)^{1/2}.$$

For problem (1) - (3) let us consider the following finite difference scheme:

$$\begin{gathered} u_{t,i}^j - \left\{ \left(1 + \tau \sum_{k=1}^{j+1} (u_{\bar{x},i}^k)^2 \right)^p u_{\bar{x},i}^{j+1} \right\}_x = f_i^j, \\ i = 1,2,...,M-1; \;\; j = 0,1,...,N-1, \end{gathered} \tag{5}$$

$$u_0^j = u_M^j = 0, \;\; j = 0,1,...,N, \tag{6}$$

$$u_i^0 = U_{0,i}, \;\; i = 0,1,...,M. \tag{7}$$

Multiplying equation (5) scalarly by u_i^{j+1}, it is not difficult to get the inequality

$$\|u^n\|^2 + \sum_{j=1}^{n} \|u_{\bar{x}}^j\|^2 \tau < C, \quad n = 1,2,...,N, \tag{8}$$

where here and below C is a positive constant independent of τ and h.

The a priori estimate (8) guarantee the stability of the scheme (5) - (7). Note, that the uniqueness of the solution of the scheme (5) - (7) can be also proved.

The main statement of the present paper can be stated as follows.

Theorem 2. *If problem (1) - (3) has a sufficiently smooth solution $U(x,t)$, then the solution $u^j = (u_1^j, u_2^j, \ldots, u_M^j)$, $j = 1,2,\ldots,N$ of the difference scheme (5) - (7) tends to the solution of continuous problem (1) - (3) $U^j = (U_1^j, U_2^j, \ldots, U_M^j)$, $j = 1,2,\ldots,N$ as $\tau \to 0$, $h \to 0$ and the following estimate is true*

$$\|u^j - U^j\| \le C(\tau + h).$$

One must note that convergence of the semi-discrete scheme for problem (1) - (3) for $0 < p \le 1$ was proven in [9]. The fully discrete analogs for $p = 1$ for of this type models and different kinds of boundary conditions are studied in [8] and in a number of other works (see, for example, [13] and references therein).

In the present work we have widened the class of nonlinearity considering case $a(S) = (1+S)^p$, $0 < p \le 1$ and investigating fully discrete finite difference schemes for that case. Various Numerical experiments using scheme (5) - (7) are carried out as well.

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