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N=2 SUPERALGEBRA AND DYNAMICAL SYMMETRIES

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Abstract. The problem of dynamical symmetry in case of Dirac Hamiltonian is considered. By help of Dirac's quantum matrix, which is conserving in any arbitrary central potential, the anticommuting algebra is constructed, which is equivalent to the Witten's $N = 2$ superalgebra. Under this algebra Dirac's quantum matrix reflects its sign. We have constructed the most general form of the generators of this superalgebra. The arbitrariness in coefficients is removed by requiring commutativity of Dirac's Hamiltonian in general central field. This requirement selects the Coulomb-like potential as the unique solution. Therefore, we prove that the symmetry under the abovementioned Witten's N=2 superalgebra unamibigously singles out the Coulomb interaction. This problem has to do with the supression of the Lamb shift.

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We consider the Dirac Hamiltonian in general central field

$$H = \boldsymbol{\alpha} \times \mathbf{p} + \boldsymbol{\beta} m + V(\boldsymbol{r}), \tag{1}$$

where $\mathbf{p}$ is a linear momentum, m is a mass, V is a potential, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are the Dirac matrixes.

In this field total momentum operator is preserved

$$\boldsymbol{J} = \boldsymbol{l} + \frac{1}{2}\boldsymbol{\Sigma},$$

where $\boldsymbol{l}$ is the operator of the orbital momentum, $\boldsymbol{\Sigma}$ is a spin's matrix

$$\boldsymbol{\Sigma} = diag(\boldsymbol{\sigma} \times \boldsymbol{\sigma})$$

.

In this field the Dirac's operator is also preserved [1]

$$K = \beta(\boldsymbol{\Sigma} \times \boldsymbol{l} + 1),$$

In the central symmetric field (1) is commuting with the Dirac's Hamiltonian $[K, H] = 0$ and has the spectrum $k = \pm(j + 1/2)$.

There is very poor knowledge about this quantum numbers.

We study the following

Problem: Whether there exists the Potential which has the spectrum, sign of which is independent of K.

Problem solution: If such operator exists it might be anticommuting with K i.e.

$$QK = -KQ, \ \{Q, K\} = 0.$$

We interested which kind of algebra will originated from this operator . Let us notate this operator by Q_1.The behavior of such kind will also has the following operator

$$Q_2 = iQ_1 \frac{K}{\sqrt{K}^2}, \tag{2}$$

The operator (2) has the following properties

$$\{Q_2, K\} = 0; \ \{Q_1, Q_2\} = 0, \ \ Q_1^2 = Q_2^2 = \tilde{H}. \tag{3}$$

Let us construct new operators

$$Q_\pm = Q_1 \pm Q_2$$

By (3) these operators have the following properties

$$Q_\pm^2 = 0; \ \{Q_+, Q_-\} = 2\tilde{H}; \ \ [Q_i, \tilde{H}] = 0, \ \ i = 1, 2.$$

The algebra which is constructed by means of this operators is called the Witten's superalgebra [2]. This algebra is so called gradual algebra. $\tilde{H}$ is a Witten's Hamiltonian. Hence, we have shown that from the operator anticommuting with the Dirac's operator the Witten's superalgebra is originated.

Let us demand that the Hamiltonian is symmetrical with respect this superalgebra. It means

$$[Q_i, H] = 0, \ i = 1, 2;$$

$$\{Q_1, K\} = 0, \ \{Q_1, H\} = 0.$$

In [3] the following theorem is proved

Theorem. *If $\boldsymbol{v}$ is the vector with respect to the momentum $\boldsymbol{l}$ i.e. $[\boldsymbol{l_i}, \boldsymbol{v_j}] = i\epsilon_{ijk}v_k$ and if $(\boldsymbol{l} \cdot \boldsymbol{v}) = (\boldsymbol{v} \cdot \boldsymbol{l}) = 0$, then the operator $(\boldsymbol{\Sigma} \cdot \boldsymbol{v})$ is anticommuting with respect to K ,or $\tilde{O}(\boldsymbol{\Sigma} \cdot \boldsymbol{v})$ is anticommuting operator with respect to K, where $\tilde{O}$ is commuting with K.*

Now, let us construct the operator Q_1.

Let us consider the vectors $\tilde{\boldsymbol{r}}$ and $\boldsymbol{p}$ and matrix γ^5 which is anticommuting with K and let us consider the general linear form

$$Q_1 = x_1(\boldsymbol{\Sigma} \cdot \tilde{\boldsymbol{r}}) + ix_2K(\boldsymbol{\Sigma} \cdot \boldsymbol{p}) + ix_3K\gamma^5 f(\boldsymbol{r}) \tag{4}$$

where x_i are the real numbers and $f(\boldsymbol{r})$ is a real function.

We admit, that (4) is commuting with the Hamiltonian

$$[Q_1, H] = (\boldsymbol{\Sigma} \cdot \tilde{\boldsymbol{r}})\{x_2 V'(\boldsymbol{r}) - x_3 f'(\boldsymbol{r})\} + 2i\beta K \gamma^5 \{x_1/|\boldsymbol{r}| - m f(\boldsymbol{r})\} = 0. \tag{5}$$

The matrix equation (5) implies

$$x_2 V'(\boldsymbol{r}) = x_3 f'(\boldsymbol{r}), \;\; x_3 m f(\boldsymbol{r}) = x_1/|\boldsymbol{r}|$$

Hence, we obtain

$$V(\boldsymbol{r}) = \frac{x_1}{x_2}\frac{1}{m|\boldsymbol{r}|}; \quad V(r) = a/|\boldsymbol{r}|.$$

Conclusion: The only potential for which Dirac's Hamiltonian has additional symmetry is the Coulomb's potential.

Now, let us consider the physical meaning of the operator (4).

It is known [4]

$$K(\boldsymbol{\Sigma} \cdot \boldsymbol{v}) = -i\beta \left(\frac{1}{2} \boldsymbol{\Sigma} \cdot [\boldsymbol{v} \times \boldsymbol{l} - \boldsymbol{l} \times \boldsymbol{v}] \right) \tag{6}$$

Putting (6) into (4) we obtain

$$Q_1 = \boldsymbol{\Sigma} \cdot \left\{ \tilde{\boldsymbol{r}} - \frac{1}{2ma} \beta [\boldsymbol{p} \times \boldsymbol{l} - \boldsymbol{l} \times \boldsymbol{p}] \right\} + \frac{i}{|\boldsymbol{r}|} K \gamma^5 \tag{7}$$

By taking nonrelativistic limits for $\beta \to 1; \gamma \to 0$ in (7) we obtain $Q_1 \to \boldsymbol{\Sigma} \cdot \boldsymbol{A}$, where

$$\boldsymbol{A} = \tilde{\boldsymbol{r}} - \frac{i}{2ma} [\boldsymbol{p} \times \boldsymbol{l} - \boldsymbol{l} \times \boldsymbol{p}].$$

The operator $\boldsymbol{A}$ is well-known Laplace-Runge-Lentz vector. So The operator (4) is Laplace-Runge-Lentz vector's projection on the direction of a particle spin.

From (7) we obtain

$$\boldsymbol{A}^2 = 1 + \left(\frac{K}{a} \right)^2 \left(\frac{H^2}{m^2} - 1 \right). \tag{8}$$

In the formula (8) all operators are commutative and A^2 is positively defined. According to this we can find ground state energy level of Hydrogen atom

$$E_0 = m \left(1 - \frac{a^2}{k^2} \right)^{1/2}. \tag{9}$$

Putting $\sqrt{k^2 - a^2} \to \sqrt{k^2 - a^2} + n - |k|$ from (9) we obtain a complete spectrum

$$E = m \left\{ 1 + \frac{a^2}{n - |k| + \sqrt{k^2 - a^2}} \right\}^{-1/2}, \tag{10}$$

where n is a natural number, $a = Ze^2$.

(10) is algebraically obtained Zommerfeld's formula. By the formula we can understand all degenerations of the Coulomb's problem in the Schrodinger's theory. We have shown that this degenerations has only the Coulomb's problem and conclude:

1) In case of $k = j + 1/2; j = l + 1/2$ we have levels $S_{1/2}, P_{3/2}, ...,$

2) In case of $k = -j - 1/2; j = l - 1/2$ we have levels $P_{1/2}, d_{3/2}, ...$

So, this degenerations suppressed the Lamb shift. Hence, we have found symmetry for which the Lamb shift is suppressed.

REFERENCES

1. Atkinson, D., Johnson, P. Quantum Field Theory, A Self-contained Course. vol. 2. *Rinton Press*, 2002.

2. Witten, E. Spontaneous symmetry breaking in SUSY quantum mechanics. *Nucl. Phys.*, **513** (1981), 1-88.

3. Khachidze, T., Khelashvili, A. Accidental smmetry operator for the Dirac equation in the coulomb potential. *Mod. Phys. Letters*, **20** (2005), 22-77.

4. Khachidze, T., Khelashvili, A. Dynamical Symmetry of the Kepler- Coulomb Problem in Classical and Quantum Mechanics. Non-Relativistic and Relativistic. *Nova Science Publishers, Inc. New York,* 2008.

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