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OPTIMAL HEDGING IN THE FINANCIAL MODEL WITH DISORDER MOMENT

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Abstract. The model of risky asset price evolution is considered and for the European contingent claim optimal in mean square sense hedging strategy is obtained.

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On the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{0\leq n\leq N}, P)$ consider the stochastic process in discrete time as evolution of the risky asset price

$$S_n = S_{n-1}\exp\{I(n<\theta)\Delta M_n^{(1)} + I(n\geq\theta)\Delta M_n^{(2)}\},\quad n=1,...,N, \tag{1}$$

where $S_0>0$ is a constant, $(M_n^{(1)}, \mathcal{F}_n), M_0^{(1)}=0$ and $(M_n^{(2)}, \mathcal{F}_n), M_0^{(2)}=0$ are independent gaussian martingales. θ is a random variable which takes values $1,2,...,N$, with known probabilities $p_i=P(\theta=i), i=\overline{1,N}$. The vector $(M^{(1)}, M^{(2)})$ is independent of θ and $I(A)$ is an indicator of $A\in\mathcal{F}$.

1. At first we find the representation for the coefficient of kurtosis of logarithmic return. It is well known, that for real financial time series this coefficient is usually positive. In our case

$$h_n = \ln\frac{S_n}{S_{n-1}} = I(n<\theta)\Delta M_n^{(1)} + I(n\geq\theta)\Delta M_n^{(2)}$$

and we have

$$K_n = \frac{E(h_n-Eh_n)^4}{(Dh_n)^2} - 3 = \frac{3P(n<\theta)P(n\geq\theta)(1-a_n)^2}{[P(n<\theta)+a_nP(n\geq\theta)]^2} > 0,$$

where $a_n = \frac{\Delta\langle M^{(1)}\rangle}{\Delta\langle M^{(2)}\rangle}$.

If for any n, $a_n\neq 1$, then the coefficient K_n is positive for each $n=1,2,...,N$.

2. Consider the financial (B,S) market and European contingent claim $f(S_N)$ ([2]), assume that interest rate $r=0$. Our aim is to find hedging strategy $(\gamma_n,\beta_n), n=\overline{1,N}$ optimal in mean square sense ([1]). So that in the special class of admissible strategies $\pi_n=(\gamma_n,\beta_n), n\in\overline{1,N}$ we minimize

$$E[f(S_N)-X_N^\pi]^2, \tag{2}$$

where X_N^π is the capital value at the terminal moment N.

We study this problem for the GF strategies, for which

$$\Delta\beta_n + \Delta\gamma_n S_{n-1} = -\Delta G_n.$$

It means, that for $\Delta G_n \geq 0$ we have some operating expenses and for $\Delta G_n \leq 0$ investment.

Under this condition for the capital process we have the following representation

$$\Delta X_n = \gamma_n \Delta S_n - \Delta G_n. \tag{3}$$

It is easy to see that

$$\begin{aligned}
&S_n = S_0 + \textstyle\sum_{k=1}^n \Delta S_k = S_0 + \sum_{k=1}^n (\Delta S_k - E(\Delta S_k/\mathcal{F}_{k-1}^S)) + \\
&\textstyle\sum_{k=1}^n E(\Delta S_k/\mathcal{F}_{k-1}^S) = m_n + A_n,
\end{aligned} \tag{4}$$

where $\mathcal{F}_n^S = \sigma\{S_k; k \leq n\}$.

From (3) and (4)

$$\Delta X_n = \gamma_n \Delta m_n + \gamma_n \Delta A_n - \Delta G_n.$$

Suppose, that $\gamma_n \Delta A_n = \Delta G_n$ and finally

$$\Delta X_n = \gamma_n \Delta m_n.$$

Consider the conditional expectation $F_n = E(f(S_N)/\mathcal{F}_n^S)$ and denote $R_0^N = E[f(S_N) - X_N]^2$. Using the equality

$$Ez^2 = (Ez)^2 + E(z - Ez)^2$$

for the value $z = f(S_N) - X_N$ we obtain

$$R_0^N = [Ef(S_N) - X_0]^2 + E[f(S_N) - Ef(S_N) - (X_N - EX_N)]^2. \tag{5}$$

The first term of (5) is minimal when $X_0 = Ef(S_N)$ and equals to zero. Consider the second term, since X_n is the martingale, then $EX_N = X_0$ and using martingale properties of F_n and m_n we obtain

$$\begin{aligned}
&E[f(S_N) - Ef(S_N) - (X_N - EX_N)]^2 \\
&= E\left(\textstyle\sum_{n=1}^N \Delta F_n - \sum_{n=1}^N \gamma_n \Delta m_n\right)^2 \\
&= E\left(\textstyle\sum_{n=1}^N (\Delta F_n - \gamma_n \Delta m_n)\right)^2 \\
&= \textstyle\sum_{n=1}^N E(\Delta F_n - \gamma_n \Delta m_n)^2.
\end{aligned} \tag{6}$$

So, that to minimize (2) we have to find minimum of (6) with respect to γ_n.

Consider

$$\gamma_n^* = \frac{E(\Delta F_n \Delta m_n / \mathcal{F}_{n-1}^S)}{E((\Delta m_n)^2 / \mathcal{F}_{n-1}^S)}, \quad n \geq 0 \tag{7}$$

and prove, that it is the optimal strategy. For that we have to show that

$$\sum_{n=1}^{N} E(\Delta F_n - \tilde{\gamma}_n \Delta m_n)^2 \geq \sum_{n=1}^{N} E(\Delta F_n - \gamma_n^* \Delta m_n)^2$$

for any $\tilde{\gamma} \in GF$.

Indeed, using the martingale property of m_n and F_n

$$\begin{aligned}
&E(\Delta F_n - \tilde{\gamma}_n \Delta m_n)^2 = E[(\Delta m_n)^2 (\tfrac{\Delta F_n}{\Delta m_n} - \tilde{\gamma}_n)^2] \\
&= E[(\Delta m_n)^2 (\tfrac{\Delta F_n}{\Delta m_n} - \gamma_n^* + \gamma_n^* - \tilde{\gamma}_n)^2] \\
&= E[(\Delta m_n)^2 [(\tfrac{\Delta F_n}{\Delta m_n} - \gamma_n^*)^2 - 2(\tfrac{\Delta F_n}{\Delta m_n} - \gamma_n^*)(\gamma_n^* - \tilde{\gamma}_n) + (\gamma_n^* - \tilde{\gamma}_n)^2]] \\
&= E[(\Delta m_n)^2 [(\tfrac{\Delta F_n}{\Delta m_n} - \gamma_n^*)^2 + (\gamma_n^* - \tilde{\gamma}_n)^2]] \\
&-2E[(\gamma_n^* - \tilde{\gamma}_n) E[(\Delta m_n)^2 (\tfrac{\Delta F_n}{\Delta m_n} - \gamma_n^*) / \mathcal{F}_{n-1}^S]] \\
&= E[(\Delta m_n)^2 [(\tfrac{\Delta F_n}{\Delta m_n} - \gamma_n^*)^2 + (\gamma_n^* - \tilde{\gamma}_n)^2]] \\
&\geq E[(\Delta m_n)^2 (\tfrac{\Delta F_n}{\Delta m_n} - \gamma_n^*)^2] = E(\Delta F_n - \gamma_n^* \Delta m_n)^2.
\end{aligned}$$

From (4) and (7) we obtain the following

Theorem. *In the model (1) of price evolution, optimal in sense (2) strategy is*

$$\gamma_n^* = \frac{E[F_n(e^{h_n} - C_n) / \mathcal{F}_{n-1}^S]}{S_{n-1} E[(e^{h_n} - C_n)^2 / \mathcal{F}_{n-1}^S]},$$

$$\beta_n^* = -C_n \sum_{i=1}^{n} \gamma_i S_{i-1} - \sum_{i=1}^{n} S_{i-1} \Delta \gamma_i,$$

where $C_n = e^{\frac{\Delta \langle M^{(1)} \rangle_n}{2}} \sum_{i=n}^{N} P_i^n + e^{\frac{\Delta \langle M^{(2)} \rangle_n}{2}} \sum_{i=1}^{n-1} Q_i^n - 1$, $P_i^n = P[(\theta = i)/\mathcal{F}_n^S)]; i > n$ *and* $Q_i^n = P[(\theta = i)/\mathcal{F}_n^S)]; i \leq n$ *are given in [3].*

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