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THE EULER BETA FUNCTION OF IMAGINARY PARAMETERS AND ITS CONNECTION WITH THE DIRAC FUNCTION *

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Abstract. We have shown that in some region where the Euler integral of the first kind diverges, the Euler formula defines a generalized function. The connection of this generalized function with the Dirac delta function is found.

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1 Introduction. It is known, that the Special and Generalized functions play a special role in theoretical and mathematical physics. We have investigated the connections between the functions, mentioned above, and have received new results. Some of them are useful in avoiding the quantum mechanical difficulties. Below will describe in detail one of the results which we have obtained.

2 Content. It is known (see, e.g. [1]) that the Euler beta function (the Euler integral of the first kind):

$$B(\alpha,\,\beta)=\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt, \qquad (1)$$
$$\mathrm{Re}\,(\alpha)>0,\ \mathrm{Re}\,(\beta)>0,$$

satisfies the Euler formula

$$B(\alpha,\,\beta)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \qquad (2)$$

where $\Gamma(z)$ is the Euler gamma function (the Euler integral of the second kind):

$$\Gamma(z)=\int_0^1 t^{z-1}\exp(-t)dt,$$

$$\mathrm{Re} z>0.$$

According to definition of the Dirac delta function (see, e.g. [2]), we assume that it is defined as weak limit of a sequence of some approximate functions: $\omega_\varepsilon(x)$

$$\delta(x)\overset{weak}{=}\lim_{\varepsilon\to 0+}\omega_\varepsilon(x), \qquad (3)$$

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where the approximate function $\omega_\varepsilon(x)$ can be constructed as (see,for example [2])

$$\omega_\varepsilon(x) = \varepsilon^{-1}\eta(\varepsilon^{-1}x), \tag{4}$$

for any bounded finite function $\eta(x)$, such that

$$\int_{-\infty}^{\infty} \eta(x)dx = 1. \tag{5}$$

In the next consideration well use functions $\eta(x) = (\pi x)^{-1}\sin x$ and $\eta(x) = [\pi(1+x^2)]^{-1}$. Another mostly symbolic but rather common definition (see, e.g. [2], [3]) considers the Dirac delta as a generalized function, which satisfies the (weak) equality

$$\int_a^b \varphi(x)\delta(x)dx = \begin{cases} 0, & 0 \notin [a,b] \\ \varphi(0), & 0 \in (a,b) \end{cases} \tag{6}$$

for any continuous function $\varphi(x)$ defined and bounded on the interval $[a,\ b]$. It was announced in [4] that the next theorem is fulfilled:

Theorem. *For* $\mathrm{Re}\alpha = \mathrm{Re}\beta = 0$ *and* $\mathrm{Im}\alpha = -\mathrm{Im}\beta = x \in \mathbb{R}$, *the Euler beta function turns into the generalized function and can be expressed through the Dirac delta function:*

$$B(i\,x,\ -i\,x) = \lim_{\varepsilon\to 0+} B(\varepsilon + ix,\ \varepsilon - ix) = 2\,\pi\,\delta\,(x). \tag{7}$$

Let us prove the next lemma before proving the theorem:

Lemma. *Let* $f(\varepsilon, x)$ *be a function of two real variables,* $\varepsilon \in (a,b)$, $a < 0 < b$, $x \in \mathbb{R}$, *which is continuous on the area of definition and has a continuous first derivative* $\frac{\partial}{\partial\varepsilon} f(\varepsilon,\ x) = f'_\varepsilon(\varepsilon,\ x)$ *and a continuous and limited second derivative*

$$\frac{\partial^2}{\partial\varepsilon^2} f(\varepsilon,\ x) = f''_{\varepsilon\varepsilon}(\varepsilon,\ x) \le M < \infty \tag{8}$$

everywhere on this area. Then

$$\lim_{\varepsilon\to 0+}\left(f(\varepsilon,x)\frac{\varepsilon}{\varepsilon^2+x^2}\right) = \pi f(0,0)\delta(x). \tag{9}$$

Proof of the Lemma. Expanding the function $f(\varepsilon,x)$ in the neighborhood of the point $(0,x)$ where $x \in \mathbb{R}$, according to the Taylor formula with remainder in the Lagrange form

$$f(\varepsilon,\ x) = f(0,\ x) + \frac{1}{1!} f'_\varepsilon(0,\ x)\varepsilon + \frac{1}{2!} f''_{\varepsilon\varepsilon}(\xi,\ x)\varepsilon^2, \quad 0 < \xi < \varepsilon,\ x \in \mathbb{R},$$

one obtains

$$f(\varepsilon,x)\frac{\varepsilon}{\varepsilon^2+x^2} = \left(f(0,x) + \frac{1}{1!} f'_\varepsilon(0,x)\varepsilon + \frac{1}{2!} f''_{\varepsilon\varepsilon}(\xi,\ x)\varepsilon^2\right)\pi\omega_\varepsilon(x),$$
$$0 < \xi < \varepsilon,\ x \in \mathbb{R}$$

where the approximate function $\omega_\varepsilon(x) = (\pi\varepsilon)^{-1}[1 + (x/\varepsilon)^2]^{-1}$ obviously satisfies (4) and (5). Calculating the weak limit $\varepsilon \to 0+$ of the both sides of the formula obtained and using the conditions of the Lemma and equality (6) one gets

$$\lim_{\varepsilon\to 0+} \int_{-\infty}^{\infty} \varphi(x) f(\varepsilon, x) \frac{\varepsilon}{\varepsilon^2 + x^2} dx$$
$$= \lim_{\varepsilon\to 0+} \int_{-\infty}^{\infty} \varphi(x) \left(f(0,x)\pi\omega_\varepsilon(x) + o(\varepsilon)\right) dx = \pi\varphi(0)f(0,0),$$

for any continuous function $\varphi(x)$.

Proof of the Theorem. Using formula (2) and the property of the gamma function

$$z\Gamma(z) = \Gamma(1+z)$$

one can write down

$$B(\varepsilon + ix, \varepsilon - ix) = \frac{\Gamma(\varepsilon + ix)\Gamma(\varepsilon - ix)}{\Gamma(2\varepsilon)} = \frac{\Gamma(\varepsilon + ix + 1)\Gamma(\varepsilon - ix + 1)}{\Gamma(2\varepsilon + 1)} \frac{2\varepsilon}{\varepsilon^2 + x^2}.$$

Now to calculate the limit in the formula (7) it is sufficient to note that the factor

$$\frac{2\Gamma(\varepsilon + ix + 1)\Gamma(\varepsilon - ix + 1)}{\Gamma(2\varepsilon + 1)}$$

satisfies the conditions of the Lemma: the function

$$f(\varepsilon,\ x) = \frac{2\Gamma(\varepsilon + ix + 1)\Gamma(\varepsilon - ix + 1)}{\Gamma(2\varepsilon + 1)} \tag{10}$$

is analytic in the point $\varepsilon = 0,\ x = 0$ of the complex plane, $f(\varepsilon, x) \in C^\infty$ and

$$\begin{aligned} f(0,\ x) &= 2\Gamma(ix+1)\Gamma(1-ix), & f(0,\ 0) &= 2,\\ f'_\varepsilon(0,\ x) &= 2f(0,\ x)\left[\mathrm{Re}\psi(1+ix) - \psi(1)\right], & f'_\varepsilon(0,\ 0) &= 0,\\ f''_{\varepsilon\varepsilon}(\xi,\ x) &= 2f(\xi,\ x)\left\{2[\mathrm{Re}\psi(\xi+ix+1) - \psi(2\xi+1)]^2\right. & &\\ &\quad \left. + [\mathrm{Re}\psi'(\xi+ix+1) - 2\psi'(2\xi+1)]\right\}, & f''_{\varepsilon\varepsilon}(\xi,\ x) &< M, \end{aligned}$$

where $\psi(z)$ is the digamma function which is determined as [5]

$$\psi(z) = \frac{d}{dz}\ln\Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}, \quad \psi(1) = -\gamma,$$

and γ is the Euler-Mascheroni constant.

Therefore, one obtains the statement to be proven (7).

Remark. Using the explicit representation (1)

$$B(\varepsilon + ix, \varepsilon - ix) = \int_0^1 dt t^{\varepsilon+ix-1}(1-t)^{\varepsilon-ix-1},$$

and applying the formula (7) one obtains

$$\lim_{\varepsilon\to 0+}\int_0^1 dt t^{\varepsilon+ix-1}(1-t)^{\varepsilon-ix-1} = 2\pi\delta(x). \tag{11}$$

On the other hand, using an insertion

$$t = \frac{1-\mathrm{th}(\xi/2)}{2}, \qquad \xi = \ln\frac{1-t}{t}, \quad d\xi = -\frac{dt}{t(1-t)},$$

it is easy to check that

$$\lim_{\lambda,\mu\to 0+}\int_{\mu}^{1-\lambda} dt t^{ix-1}(1-t)^{-ix-1} = \lim_{\alpha,\beta\to+\infty}\int_{-\alpha}^{\beta}\exp(i\xi x)d\xi = 2\pi\delta(x), \tag{12}$$

and according to the formulas (7) and (12), the integral $\int_0^1 dt t^{ix-1}(1-t)^{-ix-1}$ determines a generalized function [2]

$$B(i\,x,\,-i\,x) = \int\limits_0^1 dt t^{ix-1}(1-t)^{-ix-1} = 2\,\pi\,\delta\,(x).$$

3 Conclusions. We have defined the beta function of imaginary parameters. The function, that we have determined, is the generalized function which can be expressed by the Dirac Delta function (12). A number of rather interesting relations which follow from the results obtained above will be considered in detail in the forthcoming publications.

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