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ABOUT THE PLANE THEORY OF POROELASTICITY FOR THE BINARY MIXTURE WITH DOUBLE POROSITY ⋆

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Abstract. We consider a two-dimensional differential equations of theory of binary mixtures in the case of double porosity. The general solution of this system is represented by five analytic functions of a complex variable and solution of the Helmholtz equation. The general representation of the solution gives the opportunity to construct the analytical solutions of static boundary value problems.

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1 Main equations.

Let $Oxyz$ be a rectangular Cartesian coordinate system. Let there be a case of the plane deformation parallel to the plane Oxy for the poroelastic body consisting of binary mixture of isotropic materials. On the complex plane $z = x + iy$ the homogenous system of equations of static equilibrium has the form [1]-[8]

$$\begin{aligned} &A\Delta u_+ + 2B\partial_{\bar z}\theta - 2C\partial_{\bar z}p = 0,\\ &K\Delta p - \gamma S p = 0, \end{aligned} \qquad (1)$$

where $\bar z = x - iy$; $\partial_{\bar z} = \partial_x + i\partial_y$, $\partial_z = \partial_x - i\partial_y$, $\Delta = 4\partial_z\partial_{\bar z}$, $\partial_x \equiv \dfrac{\partial}{\partial x}$, $\partial_y \equiv \dfrac{\partial}{\partial y}$; $u_+ = (u_1^{(1)} + iu_2^{(1)}, u_1^{(2)} + iu_2^{(2)})^T$, $u_\beta^{(1)}, u_\beta^{(2)}$, $\beta = 1, 2$ are components of displacement vectors of two components of mixture; $\theta = \partial_z u_+ + \partial_{\bar z}\overline{u}_+$; $p = (p_1, p_2)^T$, p_1 and p_2 are the fluid pressures occurring respectively in the pores and fissures of the porous medium; $A = M - \lambda_5 S$, $B = M + \lambda_5 S + \Lambda$, $M = \begin{pmatrix} \mu_1 & \mu_3 \\ \mu_3 & \mu_2 \end{pmatrix}$, $\Lambda = \begin{pmatrix} \lambda_1 - \frac{\alpha\rho_2}{\rho} & \lambda_3 - \frac{\alpha\rho_1}{\rho} \\ \lambda_4 + \frac{\alpha\rho_2}{\rho} & \lambda_2 + \frac{\alpha\rho_1}{\rho} \end{pmatrix}$, $S = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, $\alpha, \lambda_1, \cdots, \lambda_5, \mu_1, \mu_2, \mu_3$ are the elastic constants characterizing mechanical properties of mixture, when $\alpha = \lambda_3 - \lambda_4$; $\rho = \rho_1 + \rho_2$, ρ_1, ρ_2 are the densities of two components of mixture; $C = \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix}$, $\beta_{11}, \cdots, \beta_{22}$ are the effective stress parameters; $K = \begin{pmatrix} k_1 & k_{12} \\ k_{21} & k_2 \end{pmatrix}$, $k_1 = \dfrac{\kappa_1}{\mu'}$, $k_2 = \dfrac{\kappa_2}{\mu'}$, $k_{12} = \dfrac{\kappa_{12}}{\mu'}$, $k_{21} = \dfrac{\kappa_{21}}{\mu'}$; μ' is fluid viscosity; κ_1 and κ_2 are

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the macroscopic intrinsic permeabilities associated with matrix and fissure porosity; κ_{12} and κ_{21} are the cross-coupling permeabilities for fluid flow at the interface between the matrix and fissure phases; $\gamma > 0$ is the internal transport coefficient and corresponds to fluid transfer rate with respect to the intensity of flow between the pore and fissures.

2 The general solution of system (1). It is possible to show that the general solution of system of equilibrium (1) is represented by means of five harmonic functions, analytic functions of a complex variable and solution of the Helmholtz equation [9]-[11].

The general solution of second equation of system (1) is written in the form

$$p = F'(z) + \overline{F'(z)} + \hat{K}\chi(z, \bar{z}), \tag{2}$$

where $F'(z) = (f'(z), f'(z))^T$, $f(z)$ is an arbitrary analytic function of a complex variable z; $\hat{K} = \begin{pmatrix} k_2 + k_{12} & 0 \\ 0 & -k_1 - k_{21} \end{pmatrix}$; $\chi(z, \bar{z}) = (\eta(z, \bar{z}), \eta(z, \bar{z}))^T$, $\eta(z, \bar{z})$ is an arbitrary solution of the Helmholtz equation

$$4\partial_z\partial_{\bar{z}}\eta - \zeta^2\eta = 0,$$

where $\zeta^2 = \dfrac{\gamma(k_1 + k_2 + k_{12} + k_{21})}{k_1k_2 - k_{12}k_{21}} > 0.$

We take the operator $\partial_{\bar{z}}$ out of the brackets in the left-hand part of the equation of system (1)

$$\partial_{\bar{z}}(2A\partial_z u_+ + B\theta - Cp) = 0. \tag{3}$$

Since (3) is a system of Cauchy-Rieman equations, we have

$$2A\partial_z u_+ + B\theta - Cp = 2(A + B)B^{-1}A\varphi'(z), \tag{4}$$

where $\varphi'(z) = (\varphi_1'(z), \varphi_2'(z))^T$, $\varphi_1(z)$ and $\varphi_2(z)$ is an arbitrary analytic function of z. By summing equation (4) with the conjugated expression we obtain

$$\theta = B^{-1}A(\varphi'(z) + \overline{\varphi'(z)}) + (A + B)^{-1}Cp. \tag{5}$$

Substituting formulas (5) into equation (4) we obtain

$$2\partial_z u_+ = A^*\varphi'(z) - \overline{\varphi'(z)} + (A + B)^{-1}Cp, \tag{6}$$

where

$$A^* = I + 2B^{-1}A, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

From formulas (2) we fined the following expression

$$p = \partial_z \left(F(z) + z\overline{F'(z)} + \frac{4}{\zeta^2}\hat{K}\partial_{\bar{z}}\chi(z, \bar{z}) \right). \tag{7}$$

And by integrating equation (6) and taking into account equation (7), we obtain

$$\begin{aligned}2u_+ = A^*\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)}+ \\ (A+B)^{-1}C\left(F(z) + z\overline{F'(z)} + \frac{4}{\zeta^2}\hat{K}\partial_{\bar z}\chi(z,\bar z)\right),\end{aligned} \tag{8}$$

where $\psi(z) = (\psi_1(z), \psi_2(z))^T$, $\psi_1(z)$ and $\psi_2(z)$is an arbitrary analytic function of z. For the complex combinations of a component of a stress tensor $\sigma_{\alpha\beta} = (\sigma^{(1)}_{\alpha\beta}, \sigma^{(2)}_{\alpha\beta})^T, \alpha, \beta = 1, 2$ we have

$$\begin{aligned}\sigma_{11} - \sigma_{22} + i(\sigma_{12} + \sigma_{21}) = 4M\partial_{\bar z}u_+, \\ \sigma_{11} + \sigma_{22} + i(\sigma_{12} - \sigma_{21}) = 2B\theta - 4\lambda_5 S\partial_z u_+ - 2Cp.\end{aligned} \tag{9}$$

Substituting expressions (7) and (8) into formulas (9), for combinations of stress tensor components we obtain the following formulas

$$\begin{aligned}\sigma_{11} - \sigma_{22} + i(\sigma_{12} + \sigma_{21}) = 2M[-\bar z\varphi''(z) + \psi'(z)+ \\ (A+B)^{-1}C(z\overline{F''(z)} + \partial_{\bar z}\partial_{\bar z}\chi(z,\bar z))], \\ \sigma_{11} + \sigma_{22} + i(\sigma_{12} - \sigma_{21}) = 2[(A - \lambda_5 SA^*)\varphi'(z)+ \\ M\overline{\varphi'(z)} - M(A+B)^{-1}C(F'(z) + \overline{F'(z)} + \hat{K}\chi(z,\bar z))].\end{aligned} \tag{10}$$

Thus, the general solution of a two-dimensional system of differential equations that describes the static equilibrium of a porous elastic medium with binary mixtures is represented by means of five analytic functions of a complex variable and solution of the Helmholtz equation. By an appropriate choice of these functions we can satisfy six independent classical boundary conditions.

3 A problem for a concentric circular ring. Let a porous elastic body with binary mixtures occupy the domain V which is bounded by the concentric circumferences L_1 and L_2 with radii R_1 and R_2 respectively ($R_1 < R_2$).

We consider the following problem ($z = re^{i\vartheta}$)

$$\sigma_{rr} + i\sigma_{r\vartheta} = \begin{cases} 0, & r = R_1, \\ 0, & r = R_2, \end{cases} \quad p = \begin{cases} p^{(1)}, & r = R_1, \\ p^{(2)}, & r = R_2, \end{cases} \tag{11}$$

where $p^{(\beta)} = (p_1^{(\beta)}, p_2^{(\beta)})^T$, $\beta = 1, 2$; $p_1^{(\beta)}$ and $p_2^{(\beta)}$ are given constant values.

For functions $f'(z)$ and $\eta(z,\bar z)$ to we receive

$$f'(z) = \frac{1}{2k_0}(a\ln z + b), \quad \eta(z,\bar z) = \frac{2}{k_0}\left(\alpha_0 I_0(\zeta r) + \beta_0 K_0(\zeta r)\right),$$

where $k_0 = k_1 + k_2 + k_{12} + k_{21}$; a, b, α_0, β_0 are certain constants; $I_0(\zeta r)$ and $K_0(\zeta r)$ are modified Bessel functions of zero order.

For the solution of a boundary value problem we use the following well-known formula

$$\sigma_{rr} + i\sigma_{r\vartheta} = 0.5[\sigma_{11} + \sigma_{22} + i(\sigma_{12} - \sigma_{21}) + (\sigma_{11} - \sigma_{22} + i(\sigma_{12} + \sigma_{21}))e^{-2i\vartheta}].$$

Further, using formula (10) we solve a problem by Muskhelishvili's method. For functions $\varphi'(z)$ and $\psi'(z)$ we receive

$$\varphi'(z) = (a_0^{(1)},\ \ a_0^{(2)})^T,\ \ \psi'(z) = (b_{-2}^{(1)},\ \ b_{-2}^{(2)})^T z^{-2},$$

where $a_0^{(\beta)}$ and $b_{-2}^{(\beta)}$, $\beta = 1, 2$ are certain constants.

By means of the constructed general solution it is possible to solve boundary value problems, when stresses and pressures on the domain boundary are given arbitrarily, but the condition that the principal vector and the principal moment of external forces are equal to zero is fulfilled.

In our opinion, problems of double porous elasticity for a binary mixtures may be of interest from theoretical and practical standpoints.

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