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HOPF BIFURCATION AND ITS COMPUTER SIMULATION FOR ONE-DIMENSIONAL MAXWELL'S MODEL

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Abstract. One-dimensional system of nonlinear partial differential equations based on Maxwell's model is considered. The initial-boundary value problem with mixed type boundary conditions is discussed. It is proved that in some cases of nonlinearity there exists critical value ψ_c of the boundary data, such that for $0 < \psi < \psi_c$ the steady state solution of the studied problem is linearly stable, while for $\psi > \psi_c$ is unstable. It is shown that when ψ passes through ψ_c then the Hopf type bifurcation may take place. The finite difference scheme is constructed. Numerical experiments agree with theoretical investigations.

Keywords and phrases: Maxwell's one-dimensional system, stationary solution, linear stability, Hopf bifurcation, computer simulation.

AMS subject classification (2010): 35B32, 35B35, 35Q61.

The present paper deals with a nonlinear model which is obtained after adding of two terms to the second equation of Maxwell's known one-dimensional system [1]. This model is also some generalization of system with two partial differential equations describing many other processes (see, for instance, [2] - [5] and references therein).

In the cylinder $[0,1]\times[0,\infty)$ let's consider the following problem [1]:

$$\frac{\partial U}{\partial t}=\frac{\partial}{\partial x}\left(V^{\alpha}\frac{\partial U}{\partial x}\right),\quad \frac{\partial V}{\partial t}=-aV^{\beta}+bV^{\gamma}\left(\frac{\partial U}{\partial x}\right)^{2}+cV^{\gamma-\alpha}\frac{\partial U}{\partial x}, \tag{1}$$

$$U(0,t)=0,\quad V^{\alpha}\frac{\partial U}{\partial x}\bigg|_{x=1}=\psi, \tag{2}$$

$$U(x,0)=U_0(x),\qquad V(x,0)=V_0(x). \tag{3}$$

Many works are dedicated to the investigation and numerical solution of (1) type models (see, for example, [6] - [12]). Here t and x are time and space variables respectively, $U=U(x,t)$, $V=V(x,t)$ are unknown functions, U_0, V_0 are given functions, $a,b,c,\alpha,\beta,\gamma,\psi$ are known positive parameters.

It is easy to check that the unique stationary solution of problem (1) - (3) is:

$$U_s(x)=\left(\frac{b}{a}\psi^2+\frac{c}{a}\psi\right)^{\frac{-\alpha}{2\alpha+\beta-\gamma}}\psi x,\quad V_s(x)=\left(\frac{b}{a}\psi^2+\frac{c}{a}\psi\right)^{\frac{1}{2\alpha+\beta-\gamma}}.$$

Introducing a designation $W=V^{\alpha}\dfrac{\partial U}{\partial x}$, after simple transformations, we get:

$$\frac{\partial W}{\partial t}=V^{\alpha}\frac{\partial^2 W}{\partial x^2}+\alpha\left(-aV^{\beta-1}+bV^{\gamma-2\alpha-1}W^2+cV^{\gamma-2\alpha-1}W\right)W,$$
$$\frac{\partial V}{\partial t}=-aV^{\beta}+bV^{\gamma-2\alpha}W^2+cV^{\gamma-2\alpha}W, \tag{4}$$

$$\left.\frac{\partial W}{\partial x}\right|_{x=0}=0,\quad W(1,x)=\psi, \tag{5}$$

$$W(x,0)=V_0^{\alpha}\frac{\partial U_0(x)}{\partial x},\quad V(x,0)=V_0(x). \tag{6}$$

The unique stationary solution of problem (4) - (6) is:

$$W_s(x)=\psi,\quad V_s(x)=\left(\frac{b}{a}\psi^2+\frac{c}{a}\psi\right)^{\frac{1}{2\alpha+\beta-\gamma}}.$$

Let $W(x,t)=W_s(x)+W_1(x)e^{\lambda t}$, $V(x,t)=V_s(x)+V_1(x)e^{\lambda t}$.

We examine the linear stability of problem (4) - (6) by linearizing (4) about the stationary solution (W_s,V_s). After some transformations we have:

$$\lambda W_1=\left(\frac{b}{a}\psi^2+\frac{c}{a}\psi\right)^{\frac{\alpha}{2\alpha+\beta-\gamma}}\frac{\partial^2 W_1}{\partial x^2}+\alpha\left(a\left(\frac{b}{a}\psi^2+\frac{c}{a}\psi\right)^{\frac{\beta-1}{2\alpha+\beta-\gamma}}\right.$$
$$\left.+b\psi^2\left(\frac{b}{a}\psi^2+\frac{c}{a}\psi\right)^{\frac{\gamma-2\alpha-1}{2\alpha+\beta-\gamma}}\right)W_1-\alpha a\psi\left(2\alpha+\beta-\gamma\right)\left(\frac{b}{a}\psi^2+\frac{c}{a}\psi\right)^{\frac{\beta-2}{2\alpha+\beta-\gamma}}V_1,$$
$$\left[\lambda+a(2\alpha+\beta-\gamma)\left(\frac{b}{a}\psi^2+\frac{c}{a}\psi\right)^{\frac{\beta-1}{2\alpha+\beta-\gamma}}\right]V_1=(2b\psi+c)\left(\frac{b}{a}\psi^2+\frac{c}{a}\psi\right)^{\frac{\gamma-2\alpha}{2\alpha+\beta-\gamma}}W_1,$$
$$\frac{\partial^2 W_1}{\partial x^2}+\eta^2 W_1=0,\quad \left.\frac{\partial W_1}{\partial x}\right|_{x=0}=W(1)=0, \tag{7}$$

where

$$\eta^2=\alpha\left(a\left(\frac{b}{a}\psi^2+\frac{c}{a}\psi\right)^{\frac{\beta-\alpha-1}{2\alpha+\beta-\gamma}}+b\psi^2\left(\frac{b}{a}\psi^2+\frac{b}{a}\psi\right)^{\frac{\gamma-3\alpha-1}{2\alpha+\beta-\gamma}}\right)$$
$$-\alpha a(2\alpha+\beta-\gamma)(2b\psi+c)\psi\left(\frac{b}{a}\psi^2+\frac{c}{a}\psi\right)^{\frac{\beta+\gamma-3\alpha-2}{2\alpha+\beta-\gamma}}$$
$$\times\left(\lambda+a(2\alpha+\beta-\gamma)\left(\frac{b}{a}\psi^2+\frac{c}{a}\psi\right)^{\frac{\beta-1}{2\alpha+\beta-\gamma}}\right)^{-1}-\lambda\left(\frac{b}{a}\psi^2+\frac{c}{a}\psi\right)^{\frac{-\alpha}{2\alpha+\beta-\gamma}}.$$

It is not difficult to show that problem (7) has nontrivial solutions if and only if

$$\eta^2=\eta_n^2=\left(n+\frac{1}{2}\right)^2\pi^2,\quad n\in Z_0.$$

For corresponding $\lambda = \lambda_n$ we have:

$$\lambda_n^2 - P_n(\psi,\alpha,\beta,\gamma,a,b,c)\lambda_n + L_n(\psi,\alpha,\beta,\gamma,a,b,c) = 0, \tag{8}$$

$$P_n(\psi,\alpha,\beta,\gamma,a,b,c) = \left(n+\frac{1}{2}\right)^2 \pi^2 \left(\frac{b}{a}\psi^2 + \frac{c}{a}\psi\right)^{\frac{\alpha}{2\alpha+\beta-\gamma}}$$
$$+a\left(\alpha+\beta-\gamma\right)\left(\frac{b}{a}\psi^2 + \frac{c}{a}\psi\right)^{\frac{\beta-1}{\alpha+\beta-\gamma}} - \alpha b\psi^2\left(\frac{b}{a}\psi^2 + \frac{c}{a}\psi\right)^{\frac{\gamma-2\alpha-1}{2\alpha+\beta-\gamma}},$$
$$L_n(\psi,\alpha,\beta,\gamma,a,b,c) = (2\alpha+\beta-\gamma)\left[a\left(n+\frac{1}{2}\right)^2 \pi^2 \left(\frac{b}{a}\psi^2 + \frac{c}{a}\psi\right)^{\frac{\beta-1}{2\alpha+\beta-\gamma}}\right. \tag{9}$$
$$\left.-\alpha a^2\left(\frac{b}{a}\psi^2 + \frac{c}{a}\psi\right)^{\frac{2(\beta-1)-\alpha}{2\alpha+\beta-\gamma}} - \alpha a b\psi^2\left(\frac{b}{a}\psi^2 + \frac{c}{a}\psi\right)^{\frac{\beta+\gamma-3\alpha-2}{2\alpha+\beta-\gamma}}\right].$$

Let's note that the stationary solution (W_s, V_s) of problem (4) - (6) is linearly stabile if and only if $Re(\lambda_n) < 0$, for all n and unstable if there is an integer m such that $Re(\lambda_m) > 0$. From (8) it can be deduced that if $2\alpha+\beta-\gamma > 0$, then stationary solution (W_s, V_s) of problem (4) - (6) is linearly stable if and only if $P_n(\psi,\alpha,\beta,\gamma,a,b,c) < 0$, for all n, i.e. if and only if

$$a\left(\gamma-\alpha-\beta\right)\left(\frac{b}{a}\psi^2 + \frac{c}{a}\psi\right)^{\frac{\beta-\alpha-1}{\alpha+\beta-\gamma}} + \alpha b\psi^2\left(\frac{b}{a}\psi^2 + \frac{c}{a}\psi\right)^{\frac{\gamma-3\alpha-1}{2\alpha+\beta-\gamma}} < \frac{\pi^2}{4}.$$

We examined the stability of the steady state solution which depends on a boundary condition $\psi > 0$. For a sufficiently small value of ψ the steady state solution is linearly stable. But when ψ passes through a critical value ψ_c, the steady state solution becomes unstable and the Hopf bifurcation may takes place [13].

In our experiment the test solutions are as follows:

$$V(x,t) = e^{-t}(\sin\pi x)^4 + 2 - x,$$

$$W(x,t) = V^{\alpha}(x,t)\left(4x^3\left(e^{-t}\left(1-x\right)^2 + \frac{\psi}{4}\right) - 2x^4\left(1-x\right)e^{-t}\right).$$

Parameters are chosen as: $a = 1, \quad b = 1, \quad c = 0, \quad \alpha = 1, \quad \beta = 3, \quad \gamma = 4, \quad \psi = 1.$

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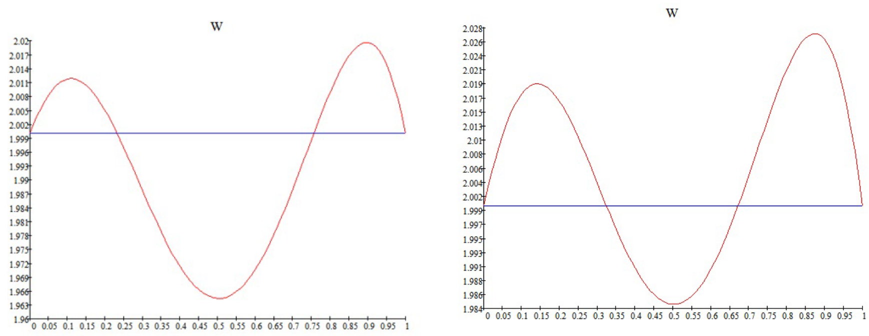

Figure 1: The solutions at $t = 0.136$ and at $t = 0.138$. The straight line is the stationary solution and curve represent the numerical solution.

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