Reports of Enlarged Sessions of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 30, 2016

ON A BOUNDARY VALUE PROBLEM FOR THE NONLINEAR NON-SHALLOW SPHERICAL SHELL ⋆

Bakur Gulua

Abstract. In this work we consider the geometrically nonlinear and non-shallow spherical shells for I.N. Vekua $N = 1$ approximation. Concrete problem using complex variable functions and the method of the small parameter has been solved.

Keywords and phrases: Stress vectors, non-shallow shells, spherical shells, small parameter.

AMS subject classification (2010): 74K25, 74B20.

In the present paper we consider the system of equilibrium equations of the two-dimensional geometrically non-linear and non-shallow spherical shells which is obtained from the three-dimensional problems of the theory of elasticity for isotropic and homogeneous shell by the method of I. Vekua [1, 2].

The displacement vector $\boldsymbol{U}(x^1, x^2, x^3)$ are expressed by the following formula [1]

$$\boldsymbol{U}(x^1, x^2, x^3) = \mathbf{u}(x^1, x^2) + \frac{x^3}{h}\mathbf{v}(x^1, x^2).$$

Here $\mathbf{u}(x^1, x^2)$ and $\mathbf{v}(x^1, x^2)$ are the vector fields on the middle surface $x^3 = 0$, $2h$ is the thickness of the shell, x^3 is a thickness coordinate $(-h \leq x^3 \leq h)$, x^1 and x^2 are isometric coordinates on the spherical surface.

Let us construct the solutions of the form [1, 3],

$$u_i = \sum_{k=1}^{\infty} \overset{k}{u}_i \varepsilon^k, \qquad v_i = \sum_{k=1}^{\infty} \overset{k}{v}_i \varepsilon^k, \quad (i = 1, 2, 3),$$

where u_i and v_i are the components of the vectors $\mathbf{u}$ and $\mathbf{v}$ respectively, $\varepsilon = \dfrac{h}{R}$ is a small parameter, R-the radius of the middle surface of the sphere..

The system of equilibrium equations of the two-dimensional geometrically nonlinear and non-shallow spherical shells may be written in the following form (approximation $N = 1$):

$$4\mu h^2 \partial_{\bar{z}} \left(\Lambda^{-1} \partial_z \overset{k}{u}_+\right) + 2(\lambda + \mu) h^2 \partial_{\bar{z}} \overset{k}{\theta} + 2\lambda h \partial_{\bar{z}} \overset{k}{v}_+ = \overset{k}{X}_+, \tag{1}$$

$$\mu h^2 \nabla^2 \overset{k}{v}_3 - 3\left[\lambda \overset{k}{\theta} + (\lambda + 2\mu) \overset{k}{v}_3\right] = \overset{k}{X}_3,$$

⋆The designated project has been fulfilled by a financial support of Shota Rustaveli National Science Foundation (Grant SRNSF/FR/358/5-109/14)

$$4\mu h^2\partial_{\bar z}\left(\Lambda^{-1}\partial_z \overset{k}{v}_+\right)+2(\lambda+\mu)h^2\partial_{\bar z}\overset{k}{\Theta}-3\mu\left(2h\partial_{\bar z}\overset{k}{v}_3+\overset{k}{v}_+\right)=\overset{k}{Y}_+, \tag{2}$$
$$\mu h\left(\nabla^2\overset{k}{u}_3+\overset{k}{\Theta}\right)=\overset{k}{Y}_3,$$
$$(k=1,2,...),$$

where λ and μ are Lame's constants, $z=x^1+ix^2, \Lambda=\dfrac{4R_0^2}{(1+z\bar z)^2}, \nabla^2=\dfrac{4}{\Lambda}\partial^2_{z\bar z}$ and

$$\overset{k}{u}_+=\overset{k}{u}_1+i\overset{k}{u}_2, \quad \overset{k}{v}_+=\overset{k}{v}_1+i\overset{k}{v}_2,$$
$$\overset{k}{\theta}=\Lambda^{-1}\left(\partial_z\overset{k}{u}_++\partial_{\bar z}\overline{\overset{k}{u}}_+\right), \quad \overset{k}{\Theta}=\Lambda^{-1}\left(\partial_z\overset{k}{v}_++\partial_{\bar z}\overline{\overset{k}{v}}_+\right).$$

Introducing the well-known differential operators

$$\frac{\partial}{\partial z}=\frac{1}{2}\left(\frac{\partial}{\partial x^1}-i\frac{\partial}{\partial x^2}\right), \quad \frac{\partial}{\partial \bar z}=\frac{1}{2}\left(\frac{\partial}{\partial x^1}+i\frac{\partial}{\partial x^2}\right).$$

$\overset{k}{X}_+, \overset{k}{Y}_+, \overset{k}{X}_3, \overset{k}{Y}_3$ are the components of external force and well-known quantity, defined by functions $\overset{0}{u}_i, ..., \overset{k-1}{u}_i, \overset{0}{v}_j, ..., \overset{k-1}{v}_j$.

The complex representation of a general solutions of systems (1) end (2) are written in the following form

$$\overset{k}{u}_+=-\frac{5\lambda+6\mu}{3\lambda+2\mu}\frac{1}{\pi}\iint_D\frac{\Lambda(\zeta,\bar\zeta)\varphi'(\zeta)d\xi d\eta}{\bar\zeta-\bar z}+\left(\frac{1}{\pi}\iint_D\frac{\Lambda(\zeta,\bar\zeta)d\xi d\eta}{\bar\zeta-\bar z}\right)\overline{\varphi'(z)}$$
$$-\overline{\psi(z)}-\frac{\lambda h}{6(\lambda+\mu)}\frac{\partial\chi(z,\bar z)}{\partial\bar z},$$
$$\overset{k}{v}_3=\chi(z,\bar z)-\frac{2\lambda h}{3\lambda+2\mu}\left(\varphi'(z)+\overline{\varphi'(z)}\right),$$
$$\overset{k}{v}_+=\frac{2(\lambda+2\mu)h^2}{3\mu}\overline{f''(z)}+\frac{1}{\pi}\iint_D\frac{\Lambda(\zeta,\bar\zeta)f'(\zeta)d\xi d\eta}{\bar\zeta-\bar z}$$
$$-\left(\frac{1}{\pi}\iint_D\frac{\Lambda(\zeta,\bar\zeta)d\xi d\eta}{\bar\zeta-\bar z}\right)\overline{f'(z)}-2h\overline{g'(z)}+i\frac{\partial\omega(z,\bar z)}{\partial\bar z},$$
$$\overset{k}{u}_3=g(z)+\overline{g(z)}-\frac{1}{\pi h}\iint_D\Lambda(\zeta,\bar\zeta)\left[f'(z)+\overline{f'(z)}\right]\ln|\zeta-z|d\xi d\eta,$$

where $\zeta=\xi+i\eta$, $\varphi(z)$,$\psi(z)$,$f(z)$ and $g(z)$ are any analytic functions of z, $\chi(z,\bar z)$ and

$\omega(z,\bar z)$ are the general solutions of the following Helmholtz's equations, respectively:

$$\Delta\chi-\kappa^2\chi=0 \qquad \left(\kappa^2=\frac{3(\lambda+\mu)}{\lambda+2\mu}h^2\right),$$
$$\Delta\omega-\gamma^2\omega=0 \qquad \left(\gamma^2=\frac{3}{h^2}\right).$$

D is the domain of the plane Ox^1x^2 onto which the midsurface S of the shell is mapped topologically.

Here we present a general scheme of solution of boundary problem when the domain D is a circle of radius r_0.

The boundary problem (in stresses) have the following form:

$$\begin{aligned}&2(\lambda+\mu)\overset{k}{\theta}+\frac{2\lambda}{h}\overset{k}{v}{}_3+2\mu\partial_{\bar z}(\Lambda^{-1}\overset{k}{u}{}_+)\left(\frac{d\bar z}{dz}\right)^2=\overset{k}{P}{}_+\quad |z|=r_0\\ &2\mu\left((\Lambda^{-1}\partial_z\overset{k}{u}{}_3+h^{-1}\overset{k}{v}{}_+)e^{i\alpha}+(\Lambda^{-1}\partial_{\bar z}\bar u^k{}_3+h^{-1}\bar v^k{}_+)e^{-i\alpha}\right)=\overset{k}{P}{}_3,\end{aligned}\tag{3}$$

$$\begin{aligned}&2(\lambda+\mu)\overset{k}{\Theta}+2\mu\partial_{\bar z}(\Lambda^{-1}\overset{k}{v}{}_+)\left(\frac{d\bar z}{dz}\right)^2=\overset{k}{Q}{}_+\quad |z|=r_0\\ &2\mu\left((\Lambda^{-1}\partial_z\overset{k}{v}{}_3)e^{i\alpha}+(\Lambda^{-1}\partial_{\bar z}\bar v^k{}_3)e^{-i\alpha}\right)=\overset{k}{Q}{}_3.\end{aligned}\tag{4}$$

Inside of the domain the analytic functions $f(z)$, $g(z)$, $\varphi(z)$ and $\psi(z)$ will have the following form:

$$f(z)=\sum_{n=1}^{+\infty}a_ne^{in\alpha},\qquad g(z)=\sum_{n=0}^{+\infty}b_ne^{in\alpha},\tag{5}$$

$$\varphi(z)=\sum_{n=1}^{+\infty}c_ne^{in\alpha},\qquad \psi(z)=\sum_{n=1}^{+\infty}d_ne^{in\alpha}.\tag{6}$$

Solutions of the Helmholtz equations $\omega(z,\bar z)$ and $\chi(z,\bar z)$ inside of the domain are represented as follows

$$\omega(z,\bar z)=\sum_{-\infty}^{+\infty}\alpha_nI_n(\gamma r)e^{in\alpha},\tag{7}$$

$$\chi(z,\bar z)=\sum_{-\infty}^{+\infty}\beta_nI_n(\nu r)e^{in\alpha},\tag{8}$$

where $I_n(\cdot)$ are Bessel's modified functions.

In the boundary conditions (3) and (4) we substitute the corresponding expressions (5)-(8), and compare the coefficients at identical degrees, we will find all coefficients [4-7].

R E F E R E N C E S

1. Vekua, I. Shell Theory: General Methods of Construction, Pitman Advanced Publishing Program. *Boston-London-Melburne*, 1985.

2. Meunargia, T. On one method of construction of geometrically and physically nonlinear theory of non-shallow shells. *Proc. A. Razmadze Math. Inst.*, **119** (1999), 133-154.

3. Vekua, I. On construction of approximate solutions of equations of shallow spherical shell. *Intern. J. Solid Structures*, **5** (1969) 991-1003.

4. Gulua, B. About one boundary value problem for nonlinear non-shallow spherical shells. *Rep. Enlarged Sess. Semin. I. Vekua Appl. Math.*, **28** (2014), 4245.

5. Gulua, B. The method of the small parameter for nonlinear non-shallow spherical shells. *Proc. I. Vekua Inst. Appl. Math.*, **63** (2013), 8-12.

6. Gulua, B. On the application of I. Vekua's method for geometrically nonlinear and non-shallow spherical shells. *Bull. TICMI*, **17**, 2 (2013), 49-55.

7. Gulua, B. On construction of approximate solutions of equations of the non-shallow spherical shell for the geometrically nonlinear theory. *Appl. Math. Inform. Mech.*, **18**, 2 (2013), 918.

Received 28.05.2016; revised 27.11.2016; accepted 20.12.2016.

Author(s) address(es):

Bakur Gulua
I. Vekua Institute of Applied Mathematics
I. Javakhishvili Tbilisi State University
University str. 2, 0186 Tbilisi, Georgia
E-mail: bak.gulua@gmail.com

Sokhumi State University
Anna Politkovskaia str. 9, Tbilisii, Georgia
E-mail: bak.gulua@gmail.com