Reports of Enlarged Sessions of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 30, 2016

MATHEMATICAL MODELING OF FILTRATION PROBLEM FOR THE MULTILAYER LIQUID-PERMEABLE HORIZONS

David Gordeziani   Tinatin Davitashvili   Teimuraz Davitashvili
Meri Sharikadze

Abstract. The present work is devoted to the analysis of some mathematical models describing a movement of subsoil waters into the soil having the non-homogeneous multilayer structure in the vertical direction. Namely the corresponding systems of two-dimensional differential equations in stationary and non-stationary cases are considered. For the first one the problem with classical and non-classical boundary conditions is stated. For numerical solution of the problem with nonlocal boundary conditions the iteration process is constructed, which allows one to reduce the solution of the initial problem to the solution of a sequence of classical Dirichlet problems. Some results of numerical calculations for the soil having two-layer structure are presented.

Keywords and phrases: Filtration problem, modeling, multilayer liquid-permeable horizons.

AMS subject classification (2010): 76N15.

1 Statement of problem. Stationary case. Let us introduce the following notetations: R^m is m-dimensional Euclidean space of points $x \equiv (x_1, \ldots, x_m)$; R^{m+1} is $(m+1)$-dimensional Euclidean space of points $(x,t) \equiv (x_1, \ldots, x_m, t)$, $t \in [0,T]$, $T = const \geq 0$; Ω is a bounded domain in R^m with the boundary Γ, $\bar{\Omega} = \Omega \cup \Gamma$; Let us consider several water-bearing horizons and the system of differential equations.

$$\begin{aligned}
&\beta_1 \Delta h_1 - (\gamma_0 + \gamma_1) h_1 + \gamma_1 h_2 = -\gamma_0 H_0, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\beta_i \Delta h_i + \gamma_{i-1} h_{i-1} - (\gamma_{i-1} + \gamma_i) h_i + \gamma_{i+1} h_{i+1} = 0, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\beta_m \Delta h_m + \gamma_{m-1} h_{m-1} - (\gamma_{m-1} + \gamma_m) h_m = -\gamma_m H_{m+1},
\end{aligned} \tag{1}$$

where $h_i = h_i(x_1, x_2)$, $x = (x_1, x_2) \subset D \subset R^2$, $\beta_i = m_i k_i$, $\gamma_i = \lambda_i / \mu_i$, are values characterizing transmission capacity (throughput) of the horizons in the horizontal and vertical directions, products $m_i \cdot k_i$ are conductivity of beds (rocks), m_i and k_i are capacity and coefficients of filtration of the i water-bearing horizons respectively, μ_j and λ_j are capacity and coefficients of filtration of the j low-penetrating horizons, respectively. It is assumed that upper and lower horizons inter-contact through layers of lower-and higher-located water-bearing horizons with constant hydraulic pressure heads H_0 and H_{m+1} (see [1]). It is also assumed that soil and liquid (fluid) are incompressible and flow is subject to Darcy's law [2]; h_i is piezometrical hydraulic pressure of the horizon i; x_3 is geometrical height, and $\Delta \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$.

Using the results from [3], we can prove that the matrix of system (1) is negatively defined, and the norm $\Phi(x) = \left[\sum\limits_{i=1}^{m} (h_i(x_1, x_2))^2\right]^{1/2}$, where $u(x)$ is a regular solution of (1), cannot achieve non-zero relative maximum for any point $x \in \Omega$. Taking into account this fact, it is easy to show that the regular solution of problem of Dirichlet for system (1) is unique. Also note that for the system (1) there exist generalized solutions for all classical boundary value problems, of course under specific restrictions for Ω, Γ and initial data. Hence, in case of classical boundary value problems, in corresponding generalized spaces, operator (1) will be coercive and therefore for these problems it will be possible to apply variational methods, methods of finite elements or finite differences. It is important to remark, that there exist rather powerful packages of the program for finite elements method for the systems similar to (1) which can by successfully applied to the considered boundary problems.

While modeling the filtration problems, penetration of admixtures into the different environments (soils, different kinds water basins) etc., there arise non-classical nonlocal boundary and initial-boundary problems.

2 Mathematical modeling of filtration problem with nonlocal boundary conditions. Let us consider the domain Ω and subdomains Ω_i $(i = 1, \cdots, k)$, where Γ is the boundary of Ω, Γ_i boundaries of Ω_i are Liapunov's manifolds and $\bar{\Omega} \supset \bar{\Omega}_1 \supset \cdots, \supset \bar{\Omega}_k$; Assume that distances between Γ and Γ_1, Γ_1 and Γ_2, etc. Γ_{k-1} and Γ_k are equal to $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_k$, $(\varepsilon_i > 0)$, respectively and Γ_i is the diffeomorphic image of Γ, i.e. $\Gamma_i = I_i(\Gamma)$, where $I_i(\cdot)$ is diffeomorphism, $i = 1, \cdots, k$. Let us consider the general nonlocal boundary condition for the system (1):

$$\begin{aligned} &\mu_0(x)\,\frac{\partial u}{\partial t}\Big|_{\Gamma} + \bar{\nu}_0(x)u(x) = \sum_{i=1}^{k} \bar{\nu}_i(x)u\left(x^{(i)}\right) + P(x) \quad x \in \Gamma, \\ &x^{(i)} \in \Gamma_i, \quad x^{(i)} = I_i(x), \quad x \in \Gamma, \end{aligned} \tag{2}$$

where $\mu_0(x)$, $u(x)$, $\nu_i(x)$, $P(x)$ are continuous functions defined on Γ; l is the unit vector, going out from the boundary point $x \in \Gamma$. Note that the problems of type (1), (2) for concrete scalar equations are widely studied.

Our aim is to reduce the solution of nonlocal boundary problem (1), (2) to the solution of a sequence of classical Dirichlet problems. For this purpose let us consider the following iteration process:

$$\begin{aligned} &\beta_1 \Delta h_1^{(p+1)} - (\gamma_0 + \gamma_1)\, h_1^{(p+1)} + \gamma_1 h_2^{(p+1)} = -\gamma_0 H_0, \\ &\beta_2 \Delta h_2^{(p+1)} + \gamma_1 h_1^{(p+1)} - (\gamma_1 + \gamma_2) h_1^{(p+1)} + \gamma_2 h_1^{(p+1)} = 0, \quad x \in \Omega, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{3}$$

$$\begin{aligned} &\beta_m \Delta h_m^{(p+1)} + \gamma_{m-1} h_{m-1}^{(p+1)} - (\gamma_{m-1} + \gamma_m) h_m^{(p+1)} = -\gamma_m H_{m+1}, \\ &\mu_0(x)\,\frac{\partial h^{(p+1)}}{\partial l}\Big|_{\Gamma} + \bar{\nu}_0(x) h^{(p+1)}(x) = \sum_{i=1}^{k} \bar{\nu}_i(x) h^{(p)}\left(x^{(i)}\right) + P(x) \quad x \in \Gamma, \\ &x^{(i)} \in \Gamma_1, \quad x^{(i)} = I_i(x), \quad x \in \Gamma, \quad p = 0, 1, 2, \cdots, \end{aligned} \tag{4}$$

where $h^{(0)}$ is an arbitrarily chosen function, but it is required that if we insert $h^{(0)}$ into the (4), the received problem should be solvable.

The following theorem is true.

Theorem. *Assume that* $\mu_0 = 0$, $\bar{\nu}_0 = E$ *(a unit operator),* $k = 1$, $\bar{\nu} \equiv const$ *and* $|\bar{\nu}_1| \leq q < 1$. *Then for the solution of the problem (4), (5)* $h^{(p)}(x_1, x_2) \to h(x_1, x_2)$ *and the following estimation is valid:*

$$\|h^{(p)} - h\| < c\bar{\nu}_1^p,$$

where $c = const$ *and doesn't depend on* $h^{(p)}$ *and* h. *To prove this theorem, the methodology offered in [4], [5] is used. Note that* $\bar{\nu}_1$ *in (4) is a coefficient of self-cleaning.*

3 Some numerical results in the case of a two-layer soil. Let us consider the equations for soil having only two permeable layers, partitioned and limited by the poorly permeable layers:

$$\begin{aligned} \sigma_1 \frac{\partial h_1}{\partial t} &= k_1 M_1 \Delta h_1 - \frac{K_0}{M_0}(h_1 - h_2) + W, \\ \sigma_2 \frac{\partial h_2}{\partial t} &= k_2 M_2 \Delta h_2 - \frac{K_0}{M_0}(h_2 - h_1) - \frac{K_{00}}{M_{00}}(h_2 - H), \end{aligned} \tag{5}$$

where σ_1, σ_2 are the coefficients of lack of saturation. K_0, K_{00} are the coefficients of filtration, M_0, M_{00} are the powers of low permeable layers.

It is evident that the operator of this system is strongly elliptic. So if one considers the classical initial-boundary problems for this equations, such problems will have a unique solution in Sobolev's spaces, of course, under the corresponding restrictions on Ω, Γ and $f_1(x, t)$, $i = 1, 2$.

The calculations were carried out for the following values of constants:
$\sigma_1 = 0.15$, $\sigma_2 = 0.12$, $k_1 = 0.54$, $k_2 = 0.63$, $M_1 = 240$, $M_2 = 124$, $K_0 = 0.35$, $K_{00} = 0.43$, $M_0 = 222$, $M_{00} = 314$, $p_0 = 150$, $s = 0.0001$, $t = 1000$, $H = 0.000267$, $x_1 = [0, 100]$, $x_2 = [0, 100]$.

$$W_s = p_0 \cdot e^{-st} \cdot 1/\left(1 + 50(x_1 - x_1^0)^2 + 50(x_2 - x_2^0)^2\right).$$

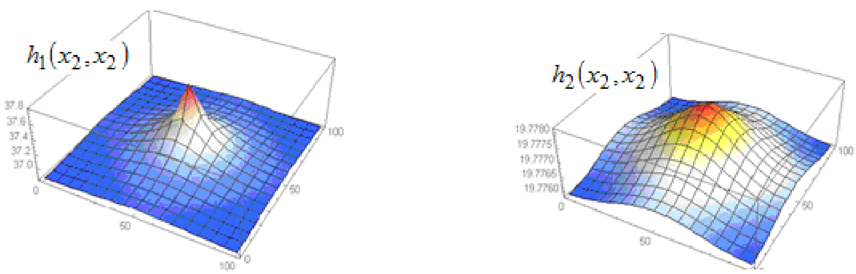

Figure 1: Stationary case, Neumann Value = 0.

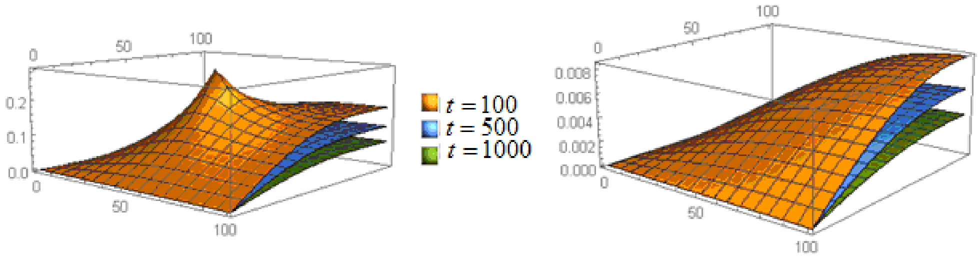

Figure 2: Nonstationary case, Neumann Value = 0.

REFERENCES

1. Polubarinova-Kochika, P.Ya., Pryazhinskaya, V.G., Emikh, V.N. Mathematical Methods in Questions of Irrigation (Russian). *Moscow, Nauka*, 1969.
2. Polubarinova-Kochika, P.Ya. Theory of the Movement of Ground Waters (Russian). *Moscow, Nauka*, 1977.
3. Bitsadze, A.V. Boundary Value Problems for Second Order Elliptic Equations (Russian). *Moscow, Nauka*, 1966.
4. Gordeziani, D.G. On One Method of Solution of Bitsadze-Samarsky boundary value problem (Russian). *Semin. I. Vekua Inst. Appl. Math. Rep.*, **26**, (1970), 39-40.
5. Gordeziani, D.G., Jioev, T.Z. About resolvability of one boundary value problem for the nonlinear equations of elliptic type (Russian). *Bulletin of AS of GSSR*, **68**, (1972), 289-292.

Received 25.05.2016; revised 17.11.2016; accepted 15.12.2016.

Author(s) address(es):

Tinatin Davitashvili
Faculty of Exact and Natural Sciences of I. Javakhishvili Tbilisi State University
University str. 2, 0186 Tbilisi, Georgia
E-mail: tinatin.davitashvili@tsu.ge

Teimuraz Davitashvili
Faculty of Exact and Natural Sciences & I. Vekua Institute of Applied Mathematics
I. Javakhishvili Tbilisi State University
University str. 2, 0186 Tbilisi, Georgia
E-mail: tedavitashvili@gmail.com

Meri Sharikadze
Faculty of Exact and Natural Sciences & I. Vekua Institute of Applied Mathematics
I. Javakhishvili Tbilisi State University
University str. 2, 0186 Tbilisi, Georgia
E-mail: meri.sharikadze@tsu.ge