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REPRESENTATION OF THE DIRAC DELTA FUNCTION IN $\mathcal{C}(R^\infty)$ IN TERMS OF THE $(1, 1, \cdots)$-ORDINARY LEBESGUE MEASURE IN R^∞ *

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Abstract. A representation of the Dirac delta function in $\mathcal{C}(R^\infty)$ in terms of the $(1, 1, \cdots)$-ordinary Lebesgue measure in R^∞ is obtained and some its properties are studied in this paper.

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1 Introduction.

The Dirac delta function(δ-function) was introduced by Paul Dirac at the end of the 1920s in an effort to create the mathematical tools for the development of quantum field theory. Later, in 1947, Laurent Schwartz gave it a more rigorous mathematical definition as a spatial linear functional on the space of test functions D (the set of all real-valued infinitely differentiable functions with compact support). Since the delta function is not really a function in the classical sense, one should not consider the value of the delta function at x. Hence, the domain of the delta function is D and its value for $f \in D$ is $f(0)$. Khuri (2004) studied some interesting applications of the delta function in statistics.

The purpose of the present paper is an introduction of a concept of the Dirac delta function in the class of all continuous functions defined in the infinite-dimensional topological vector space of all real valued sequences R^∞ equipped with Tychonoff topology.

The paper is organized as follows:

In Section 2 we present some auxiliary notions and facts which come from papers [1],[2],[3]. In Section 3 we give a representation of the Dirac delta function in $\mathcal{C}(R^\infty)$ in terms of the $(1, 1, \cdots)$-ordinary Lebesgue measure in R^∞ and consider some properties of this functional.

2 Some auxiliary notions and facts.

Definition 1. Let $(\beta_j)_{j\in N} \in [0, +\infty]^N$. We say that a number $\beta \in [0, +\infty]$ is an ordinary product of numbers $(\beta_j)_{j\in N}$ if $\beta = \lim_{n\to\infty} \prod_{i=1}^n \beta_i$. An ordinary product of numbers $(\beta_j)_{j\in N}$ is denoted by $(\mathbf{O}) \prod_{i\in N} \beta_i$.

Let $\alpha = (n_k)_{k\in N} \in (N \setminus \{0\})^N$. We set

$$F_0 = [0, n_0] \cap N,\ F_1 = [n_0+1, n_0+n_1] \cap N,\ \ldots, F_k = [n_0+\cdots+n_{k-1}+1, n_0+\cdots+n_k] \cap N, \ldots .$$

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Definition 2. We say that a number $\beta \in [0, +\infty]$ is an ordinary α-product of numbers $(\beta_i)_{i \in N}$ if β is an ordinary product of numbers $(\prod_{i \in F_k} \beta_i)_{k \in N}$. An ordinary α-product of numbers $(\beta_i)_{i \in N}$ is denoted by $(\mathbf{O}, \alpha) \prod_{i \in N} \beta_i$.

Definition 3. Let $\alpha = (n_k)_{k \in N} \in (N \setminus \{0\})^N$. Let $(\alpha)\mathcal{OR}$ be the class of all infinite-dimensional measurable α-rectangles $R = \prod_{i \in N} R_i (R_i \in \mathcal{B}(R^{n_i}))$ for which an ordinary product of numbers $(m^{n_i}(R_i))_{i \in N}$ exists and is finite. We say that a measure λ being the completion of a translation-invariant Borel measure is an α-ordinary Lebesgue measure in R^∞(or, shortly, O(α)LM) if for every $R \in (\alpha)\mathcal{OR}$ we have $\lambda(R) = (\mathbf{O}) \prod_{k \in N} m^{n_k}(R_k)$.

Lemma 1. *([1], Theorem 1. p. 217) For every $\alpha = (n_i)_{i \in N} \in (N \setminus \{0\})^N$, there exists a Borel measure μ_α in R^∞ which is O(α)LM.*

Let λ be an $(1, 1, \cdots)$-ordinary Lebesgue measure in R^∞.

Definition 4. An increasing sequence $(Y_n)_{n \in N}$ of finite subsets of the infinite-dimensional rectangle $\prod_{k \in N}[a_k, b_k] \in \mathcal{R}$ is said to be uniformly distributed in the $\prod_{k \in N}[a_k, b_k]$ if for every elementary rectangle U in the $\prod_{k \in N}[a_k, b_k[$ we have

$$\lim_{n \to \infty} \frac{\#(Y_n \cap U)}{\#(Y_n)} = \frac{\lambda(U)}{\lambda(\prod_{k \in N}[a_k, b_k[)}.$$

Lemma 2. *([3], Theorem 3.2, p.331) Let f be a continuous function on $\prod_{k \in N}[a_k, b_k]$ with respect to Tikhonov metric ρ. Then the f is Riemann-integrable on $\prod_{k \in N}[a_k, b_k]$.*

Let us denote by $\mathcal{C}(\prod_{k \in N}[a_k, b_k])$ a class of all continuous (with respect to Tikhonov topology) real-valued functions on $\prod_{k \in N}[a_k, b_k]$.

Lemma 3. *([3], Theorem 3.4, p.336) For $\prod_{i \in N}[a_i, b_i] \in \mathcal{R}$, let $(Y_n)_{n \in N}$ be an increasing family of its finite subsets. Then $(Y_n)_{n \in N}$ is uniformly distributed in the $\prod_{k \in N}[a_k, b_k]$ if and only if for every $f \in \mathcal{C}(\prod_{k \in N}[a_k, b_k])$ the following equality*

$$\lim_{n \to \infty} \frac{\sum_{y \in Y_n} f(y)}{\#(Y_n)} = \frac{(R) \int_{\prod_{k \in N}[a_k, b_k]} f(x) d\lambda(x)}{\lambda(\prod_{i \in N}[a_i, b_i])}$$

holds.

Lemma 4. *([2], Theorem 3, p.9) Let $\alpha = (n_i)_{i \in N}$ be the sequence of non-zero natural numbers and μ_α is $\mathcal{O}(\alpha)LM$. Further, let $T^{n_i} : R^{n_i} \to R^{n_i}, i \geq 1$, be a family of linear transformations with Jacobians $\Delta_i \neq 0$ and $0 < \prod_{i=1}^\infty \Delta_i < \infty$. Let $T^N : R^N \to R^N$ be the map defined by*

$$T^N(x) = (T^{n_1}(x_1, \dots, x_{n_1}), T^{n_2}(x_{n_1+1}, \dots, x_{n_1+n_2}), \dots),$$

where $x = (x_i)_{i \in N} \in R^N$. Then for each $E \in \mathcal{B}(R^N)$, we have

$$\mu_\alpha(T^N(E)) = \Big(\prod_{i=1}^\infty \Delta_i\Big) \mu_\alpha(E).$$

Lemma 5. *(Intermediate value theorem) Let f be a continuous function on $\prod_{k \in N}[a_k, b_k]$. Suppose that $\max\{f(x) : x \in \prod_{k \in N}[a_k, b_k]\} = M$ and $\min\{f(x) : x \in \prod_{k \in N}[a_k, b_k]\} = m$. Let $u \in [m, M]$. Then there is $c \in \prod_{k \in N}[a_k, b_k]$ such that $f(c) = u$.*

3 Concept of the Dirac delta function in $\mathcal{C}(R^\infty)$ and main results. Let λ be any Borel measure μ_α in R^∞. For $\epsilon > 0$, we set $a_k(\epsilon) = e^{-\frac{1}{2^k\epsilon}}/2$ and $\Delta_\epsilon = \prod_{k=1}^{\infty}[-a_k(\epsilon), a_k(\epsilon)]$. One can easily check that the equality $\lim_{\epsilon\to 0+}\mathrm{diam}(\Delta_\epsilon) = 0$ holds true.

By using Lemma 9 from Section 2, one can get the validity of the following assertion.

Theorem 1. *Let f be a continuous function on R^∞. Then the following formula*

$$\lim_{\epsilon\to O+}\frac{1}{\lambda(\Delta_\epsilon(y))}\int_{\Delta_\epsilon(y))} f(x)d\lambda(x) = f(y)$$

holds true, where $\Delta_\epsilon(y) = \Delta_\epsilon + y$ for all $y \in R^\infty$.

We have $\lambda(\Delta_\epsilon) = \prod_{k=1}^{\infty}(2a_k(\epsilon)) = e^{-\sum_{k=1}^{\infty}\frac{1}{2^k\epsilon}}$.

We set $\eta_\epsilon(x) = e^{\sum_{k=1}^{\infty}\frac{1}{2^k\epsilon}}$ if $x \in \Delta_\epsilon$ and $\eta_\epsilon(x) = 0$, otherwise. $\eta_\epsilon(x)$ is called a nascent delta function. The Dirac delta function $\delta(x)$, formally is defined by $\delta(x) = \lim_{\epsilon\to O+}\eta_\epsilon(x)$, which, of course, has no any reasonable sense.

Let f be a continuous real-valued function on R^∞. We define a Dirac delta integral as follows

$$(\delta)\int_{R^\infty}\delta(x)f(x)d\lambda(x) = \lim_{\epsilon\to O+}\int_{R^\infty}\eta_\epsilon(x)f(x)d\lambda(x).$$

We define a Dirac delta functional $\delta:\ C(R^\infty)\to R$ by $\delta(f) = (\delta)\int_{R^\infty}\delta(x)f(x)d\lambda(x)$.

The following assertion is a direct consequence of Theorem 1.

Theorem 2. *The Dirac delta functional δ is a linear functional such that $\delta(f) = f(\mathbf{0})$ for each $f \in\ C(R^\infty)$, where $\mathbf{0}$ denotes the zero of R^∞.*

By using Theorem 1 and auxiliary facts from Section 2, we get the validity of the following propositions.

Theorem 3. *For a non-zero scalar α, the infinite dimensional Dirac delta function satisfies the following scaling property*

$$(\delta)\int_{R^\infty}\delta(\alpha x)d\lambda(x) = |\alpha|^{-\infty}.$$

Theorem 4. *The infinite dimensional Dirac delta function is an even distribution, in the sense that*

$$(\delta)\int_{R^\infty}\delta(-x)f(x)d\lambda(x) = (\delta)\int_{R^\infty}\delta(x)f(x)d\lambda(x)$$

for $f \in\ C(R^\infty)$, which is homogeneous of degree -1.

Theorem 5. *(sifting property) The following equality*

$$(\delta)\int_{R^\infty}\delta(x-T)f(x)d\lambda(x) = f(T)$$

holds for $f \in\ C(R^\infty)$.

Theorem 6. *For $\epsilon > 0$, let $(Y_n(\epsilon))_{n\in N}$ be an increasing family of finite subsets of Δ_ϵ which is uniformly distributed in the Δ_ϵ. Let $f \in \mathcal{C}(R^\infty)$. Then the following formula*

$$\lim_{\epsilon\to 0+}\lim_{n\to\infty}\frac{\sum_{y\in Y_n(\epsilon)} f(y)}{\#(Y_n(\epsilon))} = f(\mathbf{0})$$

holds true.

Corollary. For $\epsilon > 0$, let $(Y_n(\epsilon))_{n\in N}$ be an increasing family of finite subsets of Δ_ϵ which is uniformly distributed in the Δ_ϵ. Let δ be Dirac delta functional defined in $\mathcal{C}(R^\infty)$. Then the following equality $\delta(f) = \lim_{\epsilon\to\mathbf{0}}\lim_{n\to\infty}\sum_{y\in Y_n(\epsilon)} f(y)/\#(Y_n(\epsilon)$ holds true for each $f \in \mathcal{C}(R^\infty)$.

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