Reports of Enlarged Sessions of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 30, 2016

# TENSOR COMPLETIONS IN VARIETIES MR-GROUPS

Mikheil Amaglobeli

**Abstract**. In the present paper some problems of the theory of the varieties of exponential MR-groups and tensor completions of MR-groups in a variety are considered.

**Keywords and phrases**: Exponential $R$-group, Lyndon $R$-group, MR-group, $\alpha$-commutator, tensor completions.

**AMS subject classification (2010)**: 20B07.

The notion of an exponential $\mathsf{R}$-group was introduced by R. lyndon in [1]. In [2] A. Myasnikov and V. Remeslennikov introduced the new category of exponential $\mathsf{R}$-groups (MR-groups) as a natural generalization of an $\mathsf{R}$-module to the noncommutative case. Below, we recall the basic definitions borrowed from [1, 2].

## 1 Definition of an exponential MR-groups.

Fix to the rest of the paper an arbitrary associative ring $\mathsf{R}$ with a unit and a group $G$. Let an action of the ring $\mathsf{R}$ on $G$ be given which is a mapping $G \times \mathsf{R} \to G$. The result of the action of $\alpha \in \mathsf{R}$ on $g \in G$ will be written as $g^\alpha$. Consider the axioms:

(i) $g^1 = g$, $g^0 = e$, $e^\alpha = e$;

(ii) $g^{\alpha+\beta} = g^\alpha \cdot g^\beta$, $g^{\alpha\beta} = (g^\alpha)^\beta$;

(iii) $(h^{-1}gh)^\alpha = h^{-1}g^\alpha h$;

(iv) $[g,h] = 1 \Longrightarrow (gh)^\alpha = g^\alpha h^\alpha$ (MR-axiom).

**Definition 1.** The group $G$ is called an **exponential** $\mathsf{R}$-group (or **R-group**) after Lyndon if an action of the ring $R$ on $G$ satisfying axioms (i)–(iii) is given.

**Definition 2.** The group $G$ is called an exponential **R-group** (or **MR-group**) if an action of the ring $\mathsf{R}$ on $G$ satisfies axioms (i)–(iv).

Then $\mathsf{R}$ is called a **ring** of **scalars** of the group $G$. Let $\mathfrak{L}_\mathsf{R}$ and $\mathfrak{M}_\mathsf{R}$ be the classes of all exponential $\mathsf{R}$-groups after Lyndon and all MR-group, $\mathfrak{L}_\mathsf{R} \supseteq \mathfrak{M}_\mathsf{R}$. There exist Abelian Lyndon $\mathsf{R}$-groups which are not $\mathsf{R}$-modules (see [3], where the structure of the free Abelian $\mathsf{R}$-group was studied in detail).

Most of natural examples of exponential groups belongs to the class $\mathfrak{M}_\mathsf{R}$:

1) an arbitrary group is a $\mathbb{Z}$-group;

2) an Abelian divisible group from $\mathfrak{L}_{\mathbb{Q}}$ is a $\mathbb{Q}$-group;

3) a group of the period $n$ is a $\mathbb{Z}/n\mathbb{Z}$-group;

4) a module over the ring $\mathsf{R}$ is an Abelian $\mathsf{MR}$-group;

5) free Lyndon $\mathsf{R}$-groups are $\mathsf{MR}$-group;

6) the exponential nilpotent $\mathsf{R}$-groups over the binomial ring $\mathsf{R}$ introduced by P. Hall in [4] are $\mathsf{MR}$-groups;

7) an arbitrary pro-$p$-group is a $\mathbb{Z}_{p^\infty}$-group over a ring of integer $p$-adic numbers $\mathbb{Z}_{p^\infty}$;

8) an arbitrary profinite group is a $\widehat{\mathbb{Z}}$-group, where $\widehat{\mathbb{Z}}$ is the total completion of $\mathbb{Z}$ in the profinite topology;

(9) complex-valued (real) nilpotent Lie groups are $\mathbb{G}$- (-$\mathbb{R}$)-groups.

A systematic stufy of $\mathsf{MR}$-group was initiated in [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]. Results obtained in these papers have turned out to be very useful in solving well-known problems of Tarski.

**Definition 3.** A homomorphism of $\mathsf{R}$-groups $\varphi : G_1 \to G_2$ is called an *R-homomorphism* if

$$(g^\alpha)^\varphi = (g^\varphi)^\alpha, \quad g \in G, \quad \alpha \in \mathsf{R}\,.$$

Let $G$ be an $\mathsf{R}$-group. Let us introduce the following designations: $x^{\mathsf{R}} = \{x^\alpha \mid \alpha \in \mathsf{R}\}$, $X^{\mathsf{R}} = \bigcup\limits_{x \in X} x^{\mathsf{R}}$, $X \subseteq G$.

**Definition 4.** The subgroups $H \leq G$ is called an **R-subgroup** if $H^{\mathsf{R}} = H$. The subgroup $H$ is **R-generated** by a set $X \subseteq G$ if $H$ is the least $\mathsf{R}$-subgroup of $G$ which contains $X$. In this situation we have $H = \langle X \rangle_{\mathsf{R}}$.

**Definition 5.** For $g, h \in G$ and $\alpha \in \mathsf{R}$, the element $(g,h)_\alpha = h^{-\alpha} g^{-\alpha} (gh)^\alpha$ is called the **$\alpha$-commutator** of the elements $g$ and $h$.

Clearly, $(gh)^\alpha = g^\alpha h^\alpha (g,h)_\alpha$ and $G \in \mathfrak{M}_{\mathsf{R}} \Longleftrightarrow ([g,h] = e \longrightarrow (g,h)_\alpha = e)$. This equivalence leads to the definition of an $\mathfrak{M}_{\mathsf{R}}$-ideal.

**Definition 6.** A normal $\mathsf{R}$-subgroup $H \trianglelefteq G$ is called an **$\mathfrak{M}_{\mathsf{R}}$-ideal** if for $g \in G$, $h \in H$ and $\alpha \in \mathsf{R}$

$$[g,h] \in H \Longrightarrow (g,h)_\alpha \in H.$$

**Proposition.** *Let $G \in \mathfrak{M}_R$. Then*

1. *if $\varphi : G \to G'$ is an $R$-homomorphism of groups from $\mathfrak{M}_R$, then* $\operatorname{Ker} \varphi$ *is an $\mathfrak{M}_R$-ideal in $G$;*

2. *if $H$ is an $\mathfrak{M}_R$-ideal in $G$, then $G/H \in \mathfrak{M}_R$.*

**2 Tensor completions in varieties.** All the necessary information about the varieties of MR-groups see in [15].

**Definition 1.** Let $\mathfrak{N}_{\mathsf{R}}$ be the variety of MR-groups given by the set of words $W$, let $R \subseteq \mathsf{S}$, $G$ be a group from $\mathfrak{N}_{\mathsf{R}}$. The group $G_W^{\mathsf{S}} \in \mathfrak{N}_{\mathsf{S}}$ is called a **tensor S-completion of $G$ in the variety $\mathfrak{M}_{\mathsf{S}}$**, if there exists an R-homomorphism $\lambda : G \to G_W^{\mathsf{S}}$ such that $G_W^{\mathsf{S}} = \langle \lambda(G) \rangle_{\mathsf{S}}$ and for any group $H$ from $\mathfrak{N}_{\mathsf{S}}$ and any R-homomorphism $\varphi : G \to H$ there exists a S-homomorphism $\psi : G_W^{\mathsf{S}} \to H$ that closes the diagram

$$\begin{tikzcd} G \arrow[r, "\lambda"] \arrow[d, "\forall\varphi"'] & G^{\mathsf{S}} \arrow[dl, dashed, "\exists\psi"] \\ H & \end{tikzcd} \qquad (\lambda\psi = \varphi)$$

and makes it commutative.

Note that further we consider only the situation in which $\mathsf{R} \to \mathsf{S}$ is an embedding and therefore does not participate in the definition and notation. This restriction is not essential and is made only to simplify the notation.

**Theorem 1.** *Let $G \in \mathfrak{N}_{R}$. Then tensor S-completion $G_W^{S}$ with respect to $\mathfrak{N}_{S}$ exists and*

$$G_W^{S} \cong G^{S}/W(G^{S}).$$

**Theorem 2.** *Let $R \subseteq S$ be rings and let $F_{W,R}(X)$ be a free group in the variety $\mathfrak{N}_{R}$. Then $(F_{W,R}(X))_W^{S}$ is a free group in the variety $\mathfrak{N}_{S}$, i.e.*

$$(F_{W,R}(X))_W^{S} \cong F_{W,S}(X).$$

In [1] it is stated that tensor completions of abelian groups are abelian groups. In the general case a tensor completion in the category of all exponential groups is obtained by means of free constructions and therefore, as a rule, contains free subgroups in the non-commutative case.

**Theorem 3.** *If $G$ us a class-2 of nilpotent MR-groups, then its tensor-completion $G^S$ is the class-2 nilpotent MS-group.*

### REFERENCES

1. Myasnikov, A.G., Remeslennikov V.N. Degree groups. I. Foundations of the theory and tensor completions (Russian). *Sibirsk. Mat. Zh.* **35**, 5 (1994), 1106-1118; English translation *Siberian Math. J.*, **35**, 5 (1994), 986-996.

2. Lyndon, R.C. Groups with parametric exponents. *Trans. Amer. Math. Soc.*, **96** (1960), 518-533.

3. Baumslag, G. Free Abelian $X$-groups. *Illinois J. Math.*, **30**, 2 (1986), 235-245.

4. Hall, P. Nilpotent Groups. *Canad Math. Congress. Edmonton*, 1957.

5. Myasnikov, A.G., Remeslennikov, V.N. Exponential groups. II. Extensions of centralizers and tensor completion of CSA-groups. *Internat. J. Algebra Comput.*, **6**, 6 (1996), 687-711.

6. Baumslag, G., Myasnikov, A., Remeslennikov, V. Discriminating completions of hyperbolic groups. Dedicated to John Stallings on the occasion of his 65th birthday. *Geom. Dedicata*, **92** (2002), 115-143.

7. Amaglobeli, M. G., Bokelavadze, T.Z. Power groups. Groups which are accurate at tensor completion. *Vestn. Omsk. Univ.*, **2009**, 2 (2009), 35-46.

8. Amaglobeli, M.G. On the permutability of a functor of tensor completion with principal group operations. *Appl. Math Inform. Mech.*, **15**, 1 (2010), 3-10, 54.

9. Amaglobeli, M., Bokelavadze, T. Abelian and nilpotent varieties of power groups. *Georgian Math. J.*, **18**, 3 (2011), 425-439.

10. Amaglobeli, M.G., Remeslennikov, V.N. Free nilpotent $R$-groups of class 2 (Russian). *Dokl. Akad. Nauk* **443**, 4 (2012), 410-413; English translation *Dokl. Math.* **85**, 2 (2012), 236-239.

11. Amaglobeli, M. Free products in the category of $a$-groups over a ring $A$ (Russian). *Sovrem. Mat. Prilozh.*, **74** (2011); English translation *J. Math. Sci. (N. Y.)* **186**, 5 (2012), 687-693.

12. Amaglobeli, M.G. Power groups (Russian). *Sovrem. Mat. Prilozh.*, **75** (2011); English translation *J. Math. Sci. (N. Y.)*, **186**, 6 (2012), 811-865.

13. Amaglobeli, M. Tensor completion in the category of $R$-groups over a ring $R$. (Russian). *Sovrem. Mat. Prilozh.* **80**, 2012, Part 1; English translation *J. Math. Sci. (N. Y.)* **193**, 3 (2013), 353-358.

14. Amaglobeli, M.G., Remeslennikov, V.N. Extension of a centralizer in nilpotent groups (Russian). *Sibirsk. Mat. Zh.*, **54**, 1 (2013), 8-19.

15. Amaglobeli, M. Varieties of exponential MR-groups. *Rep. Enlarged Sess. Semin. I. Vekua Inst. Appl. Math.*, **29** (2015).

16. Amaglobeli, M., Remeslennikov, V. Algorithmic problems for class-2 nilpotents MR-groups. *Georgian Math. J.*, **22**, 4 (2015), 441-449.


Received 24.05.2016; revised 17.11.2016; accepted 13.12.2016.

Author(s) address(es):

Mikheil Amaglobeli
I. Vekua Institute of Applied Mathematics
I. Javakhishvili Tbilisi State University
University str. 2, 0186 Tbilisi, Georgia
E-mail: mikheil.amaglobeli@tsu.ge