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ON ONE BOUNDARY VALUE PROBLEM OF THE GENERALIZED ANALYTIC VECTORS ⋆

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Abstract. The Dirichlet-type problem for one quasi-linear elliptic system is investigated.

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In the work of Bojarski [1] it was shown that the methods of generalized analytic functions [2] admit further generalization to the case of the first order elliptic systems the complex form of which is the following

$$\partial_{\bar{z}}w - Q(z)\partial_z w + Aw + B\bar{w} = 0, \tag{1}$$

where $\partial_z \equiv \frac{1}{2}(\partial_x - i\partial y), Q(z), A(z), B(z)$ are given square matrices of order $n, Q(z)$ has special quasi-diagonal form.

Hile [3] noted that what appears to be the essential property of the elliptic systems on the plane for which one can obtain a useful extension of the analytic function theory is the self-commuting property of the variable matrix Q

$$Q(z_1)Q(z_2) = Q(z_2)Q(z_1) \tag{2}$$

for any two points z_1, z_2 of the complex plane C.

Following Hile if $A = B = 0$ and Q is self-commuting in C and if $Q(z)$ has eigenvalues less than 1 then the system (1) is called the generalized Beltrami system. The solution of such system is called Q-holomorphic vector. Under the solution in some domain D we mean the so-called regular solution - $w(z) \in L_2(D)$ whose generalized derivatives $w_{\bar{z}}, w_z$ belong to $L_r(D')$, $r > 2$ where $D' \subset D$ is an arbitrary closed subset and this system is to be satisfied almost everywhere in D.

The matrix valued function $\Phi(z)$ is a generating solution of the generalized Beltrami system if it satisfies the following properties ([3]):

(i) $\Phi(z)$ is a C^1-solution of the generalized Beltrami system in C;

(ii)$\Phi(z)$ is self-commuting and commutes with Q in C;

(iii) $\Phi(t) - \Phi(z)$ is invertible for all z, t in C, $z \neq t$;

(iv) $\partial_z \Phi(z)$ is invertible for all z in C.

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We will call the matrix $V(t,z) = \partial_z \Phi(t)[\Phi(t) - \Phi(z)]^{-1}$ the generalized Cauchy kernel for the system (1).

Introduce some classes of Q-holomorphic vectors. Let be D a bounded domain with a sufficiently smooth boundary. We say, that the Q-holomorphic vector $\Phi(z)$ belongs to the class $E_p(D,Q),\ p > 1$ if the vector $\Phi(z)$ is representable by the generalized Cauchy-Lebesgue type integral

$$\Phi(z) = \frac{1}{2\pi i} \int\limits_{\Gamma} V(t,z) d_Q t \varphi(t) \tag{3}$$

in the domain D, where $\varphi(t) \in L(\Gamma), d_Q t = I dt + Q d\bar{t}, I$ is an identity matrix.

We need the following result in the sequel.

Theorem 1. *Let $\Phi(z) \in L_p(\Gamma),\ p > 1$. Then the generalized Cauchy-Lebesgue integral (3) belongs to the class $L_s(\bar{D}),\ s = 2p$. Moreover, the following inequality*

$$\|\Phi\|_{L_s^n(\bar{D})} \leq M(p,D) \|\varphi\|_{L_p^n} \tag{4}$$

holds. (The notation $A \in K$, where A is a matrix and K is some class of the functions, means that every element $A_{\alpha\beta}$ of A belongs to K. If K is some linear normed space with the norm $\|\cdot\|$ then $\|A\|_k = max_{\alpha\beta} \|A_{\alpha\beta}\|_k$)

Let now Γ be a union of simple closed non-intersecting Liapunov-smooth curves bounding finite or infinite domain D. Consider the boundary value problem for the system (1) the so-called modified Dirichlet problem [4]

$$Re[w(t)] = f(x) + a(t), \quad t \in \Gamma, \tag{5}$$

$$Im[w(z_0)] = c_0, \quad z_0 \in D, \tag{6}$$

where $f(x)$ is a given real vector on Γ, c_0, given real constant vector, z_0 is an arbitrary fixed point of the domain $D, a(t) = (a_0, a_1, \ldots, a_m)$ on Γ is a piecewise constant vector which is not defined beforehand. The vector $a(t)$ is completely defined by the problem itself if one of its values is arbitrarily fixed. We assume that $a_0 = 0$ in what follows.

Every Q-holomorphic vector $\Phi(z)$ of the class $E_p(D,Q)$ admits the following representation

$$\Phi(z) = \frac{1}{\pi i} \int_{\Gamma} V(t,z) d_Q t \mu(t) + ic \tag{7}$$

where $\mu(t) \in L_p(\Gamma, \rho)$ is a real vector. The vector $\mu(t)$ is defined on $\Gamma_j,\ j \geq 1$ identically to within the constant vector, $\mu(t)$ on Γ_0 and the constant vector c are defined by the vector $\Phi(z)$ uniquely.

Using the representation (7) and the analog of the Sokhotsky-Plemelj formula for the integral (7) and the boundary condition (5) we obtain the system of Fredholm type integral equations

$$[(I + M)\mu](t) = f(x) + a(t). \tag{8}$$

The corresponding homogeneous equation of the equation (8) has nm independent solutions. Therefore the inhomogeneous equation (8) is solvable if and only if the right-hand side of this equation satisfies some nm conditions. For the solution of the problem the constant vectors a_j should be chosen so that these conditions will be fulfilled.

Following Muskhelishvili [4] instead of the equation (8) consider its equivalent equation

$$(K\mu)(t) = f(t_0), \tag{9}$$

where

$$(K\mu)(t) = [(I+M)\mu](t) - \int_\Gamma K(t,t_0)\mu(t)ds, \; k(t,t_0) = \begin{cases} \rho_j(t)I, & t,t_0 \in \Gamma_j \\ 0, & \textit{in all other ceses.} \end{cases} \tag{10}$$

$\rho_j(t)$ denotes a real continuous function given on Γ_j, $(j = 1,\ldots,m)$ satisfying the condition

$$\int_{\Gamma_j} \rho_j(t)ds \neq 0. \tag{11}$$

It can be proved that the homogeneous equation (9) has not non-trivial solutions. From here and by virtue of Fredholm theorem it follows that inhomogeneous equation (9) has unique solution $\mu(t)$ which is a solution of the initial equation (8). The vectors a_j are completely defined values, namely $a_j = \int_{\Gamma_j} \mu ds$.

We get that the linear bounded operator K in (9) is an invertible operator in the Banach space L_p^n, $p > 1$. Therefore

$$\|\mu\|_{L_s^n(\Gamma)} \leq A(p,\Gamma)\|f\|_{L_p^n(\Gamma)} \; where \; A(p,\Gamma) = \|K^{-1}\|_{L_p^n(\Gamma)}. \tag{12}$$

Applying the estimation given in Theorem 1 the following estimation

$$\|\Phi\|_{L_s^n(\bar{D})} \leq B(p,D)\|f\|_{L_p^n(\Gamma)} + \sum_{i=1}^n |c_{0,i}|(mD)^{\frac{1}{p}} \tag{13}$$

holds, where Φ is a Q-holomorphic vector which is the solution of the boundary problem (5)-(6), mD is the measure of the domain D.

Consider now the nonlinear differential system in the domain D which has the following complex form

$$\frac{\partial w}{\partial \bar{z}} - Q(z)\frac{\partial w}{\partial z} = F(.,w) \quad where \quad F(.,w) \; is \; a \;\; n \times 1 \; - \; matrix. \tag{14}$$

Investigate the boundary value problem (5)-(6) for the system (14), $f(t) \in L_p(\Gamma)$, $p > 1$ is a given vector on Γ:

Find the vector $w(z) = (w_1,\ldots,w_n) \in W_p^1(D)$, $p > 2$ from the class $E_p(D,Q) + C(\bar{D})$ satisfying (14) almost everywhere on Γ.

With respect to F the following conditions are supposed to be fulfilled

1) $F(z,w)$ is measurable with respect to z for every fixed w;

2) F satisfies the Lipshitz condition (15)

$$|F(z,w_1)-F(z,w_2)| \leq L|w_1-w_2|, \quad L>0;$$

3) $F(z,0) \in L_S(\bar{D})$, where $s=2p$.
From these conditions it follows that $F(z.w) \in L_s(\bar{D})$ for every vector $w \in L_s(\bar{D})$.
Using the above obtained results we get the following theorem.

Theorem 2. *If the right-hand side F of the system (14) satisfies the conditions (15) then in case of a sufficiently small Lipshitz constant L the boundary value problem (5)-(6) for every given $f(t) \in L_p(\Gamma)$, $p>1$ and c_0 has a unique solution of the class $E_p(D,Q)+C(\bar{D})$. The piecewise constant vector $a(t)$ is completely defined by the problem itself if one of its values is arbitrarily fixed.*

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