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ON THE NUMBER OF REPRESENTATIONS OF A POSITIVE INTEGER BY BINARY FORMS BELONGING TO GENERA CONSISTING OF TWO CLASSES

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Abstract. Using the formulas for the average number of representations of a positive integer we show the existence of binary forms which belong to two-class genera, but for which the number of representations of natural numbers is equal to the average number of representations by the corresponding genus.

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Let $r(n;f)$ denote the number of representations of a natural number n by a positive integral quadratic form $f = f(x_1, x_2, \ldots, x_s)$. Finding an exact formula for the function $r(n;f)$ means (i) constructing a singular series $\rho(n;f)$ that corresponds to the quadratic form f, (ii) finding its sum, and (iii) constructing function $X(\tau;f) = \sum\limits_{n=1}^{\infty} \nu(n;f)\, Q^n$ $(Q = e^{2\pi i\tau})$ which is regular for $\operatorname{Im}\tau > 0$, so that the following equality holds

$$r(n;f) = \rho(n;f) + \nu(n;f). \tag{1}$$

It is known that $\vartheta(\tau;f) = 1 + \sum\limits_{n=1}^{\infty} r(n;f)\, Q^n$. Therefore if the function

$$E(\tau;f) = 1 + \sum_{n=1}^{\infty} \rho(n;f)\, Q^n$$

is regular for $\operatorname{Im}\tau > 0$, then the arithmetic equality (1) is evidently equivalent to the functional one

$$\vartheta(\tau;f) = E(\tau;f) + X(\tau;f).$$

Let $F(\tau;f)$ denote the theta-series of the genus containing a primitive integral quadratic form f. Siegel [1] proved that if the number of variables of a quadratic form f (both positive-definite and indefinite) is $s > 4$, then

$$F(\tau;f) = E(\tau;f), \tag{2}$$

where $E(\tau;f)$ is the Eisenstein series.

Later Ramanathan [2] proved that for any primitive integral quadratic form f with $s \geq 3$ variables (except for zero forms with variables $s = 3$ and zero forms with variables $s = 4$ whose discriminant is a perfect square), there is a function $E(\tau, z; f)$ which is called Eisenstein-Siegel series and which is regular for any fixed τ when $\operatorname{Im}\tau > 0$ and $\operatorname{Re} z > 2 - \frac{s}{2}$, analytically extendable in a neighborhood of $z = 0$, and that

$$F(\tau; f) = E(\tau, z; f)\big|_{z=0}. \tag{3}$$

For $s > 4$ the function $E(\tau, z; f)\big|_{z=0}$ coincides with the function $E(\tau; f)$, and the formula (3) with Siegel's formula (2).

In [3] we proved that the function $E(\tau, z; f)$ is analytically extendable in a neighborhood of $z = 0$ also in the case where f is any nonzero integral binary quadratic form (both positive-definite and indefinite) and that

$$F(\tau; f) = \frac{1}{2}\, E(\tau, z; f)\big|_{z=0}. \tag{4}$$

Moreover, convenient formulas are obtained for calculating the values of the function $\rho(n; f)$ in the case where f is any positive integral form of variables $s \geq 2$ (see [4], [5], [6]).

It follows from (4) that half of "the sum of a generalized singular series" that corresponds to the binary quadratic form is equal to the average number of representations of a natural number by the genus containing this quadratic form.

In particular, if a quadratic form belongs to a one-class genus, then for natural n

$$r(n; f) = \frac{1}{2}\, \rho(n; f). \tag{5}$$

In this paper we show the existence of positive integral binary forms which belong to two-class genera, but for which equality (5) is true for some natural n.

Having calculated the values of $\frac{1}{2}\rho(n; f)$ by the formulas from [5], we obtain the formulae for the number of representations of natural numbers by these forms.

Let $f = ax^2 + bxy + cy^2$ be a positive-definite binary quadratic form of discriminant d, so that $d = b^2 - 4ac$. We denote the number of representations of n by the form f by $r(n; a, b, c)$. The set of forms with discriminant -63 splits into two genera of forms and each genus consists of two classes of forms which are respectively

$$f_1 = 2\,x^2 + xy + 8\,y^2, \quad f_2 = 2\,x^2 - xy + 8\,y^2$$

and

$$f_3 = x^2 + xy + 16\,y^2, \quad f_4 = 4\,x^2 + xy + 4\,y^2.$$

It is obvious that $r(n; 2, 1, 8) = r(n; 2, -1, 8)$ and $\rho(n; 2, 1, 8) = \rho(n; 2, -1, 8)$.

Thus by Siegel's theorem

$$r(n; 2, 1, 8) = r(n; 2, -1, 8) = \frac{1}{2}\, \rho(n; 2, 1, 8).$$

The function $\rho(n; 2, 1, 8)$ can be calculated by the formulas from [5]. So we have

Theorem 1. *Let $n = 2^\alpha\, 3^\beta\, 7^\gamma\, u$, where $(u, 42) = 1$. Then we have*

$$\begin{aligned} r = (n; 2, 1, 8) = r(n; 2, -1, 8) \\ &= \frac{1}{4}\,(\alpha+1)\Big(1 - \Big(\frac{n}{3}\Big)\Big)\Big(1 + \Big(\frac{7^{-\gamma} n}{7}\Big)\Big) \sum_{\nu\mid u} \Big(\frac{-7}{\nu}\Big) \quad \textit{for} \quad \beta = 0, \\ &= (\alpha+1)\Big(1 + \Big(\frac{7^{-\gamma} n}{7}\Big)\Big) \sum_{\nu\mid u} \Big(\frac{-7}{\nu}\Big) \quad \textit{for} \quad 2\,|\,\beta,\ \ \beta > 0, \\ &= 0 \quad \textit{for} \quad 2 \dagger \beta. \end{aligned}$$

Here $\big(\frac{7^{-\gamma} n}{7}\big)$, $\big(\frac{-7}{\nu}\big)$ *are Legendre-Jacobi symbols.*

Let $2 \dagger m$

$$x^2 + xy + 16\, y^2 = 2\, m.$$

Then $2 \dagger x$, $2 \dagger y$ or $2\,|\,x$, $2 \dagger y$,

$$4\, X^2 + XY + 4\, Y^2 = 2\, m,$$

where

$$x = 2\, X + \frac{1}{2}\, Y, \quad y = -\frac{1}{2}\, Y; \quad X = \frac{x+y}{2}\,, \quad Y = -2\, y$$

or

$$x = 2\, Y, \quad y = \frac{1}{2}\, X; \quad X = 2\, y, \quad Y = \frac{1}{2}\, x.$$

Thus we have

$$r(2m; 1, 1, 16) = r(2m; 4, 1, 4), \quad \text{where} \quad 2 \dagger m.$$

Let

$$4\, x^2 + xy + 4\, y^2 = 9\, n.$$

Then $x \equiv y \pmod 3$,

$$2\, X^2 + 3\, XY + 2\, Y^2 = n,$$

where

$$x = X + 2\, Y, \quad y = -2\, X - Y; \quad X = -\frac{x+2y}{3}\,, \quad Y = \frac{2x+y}{3}\,.$$

The binary form $2\, x^2 + 3\, xy + 2\, y^2$ belongs to the one class genus containing the reduced form $x^2 + xy + 2\, y^2$.

Thus we have

$$r(9\, n; 1, 1, 16) = r(9\, n; 4, 1, 4) = r(n; 1, 1, 2) = \frac{1}{2}\, \rho(n; 1, 1, 2).$$

Having calculated the values of the functions $\rho(2m; 4, 1, 4)$ and $\rho(n; 1, 1, 2)$, we get

Theorem 2. *Let $n = 2^\alpha\, 3^\beta\, 7^\gamma\, u$, where $(u, 42) = 1$. Then we have*

$$r = (n; 1, 1, 16) = r(n; 4, 1, 4) = \frac{1}{2}\left(1 - \left(\frac{u}{3}\right)\right)\left(1 + \left(\frac{u}{7}\right)\right)\sum_{\nu | u}\left(\frac{\nu}{7}\right)$$

$$\textit{for} \quad \beta = 0, \quad \alpha = 1,$$

$$= (\alpha + 1)\left(1 + \left(\frac{u}{7}\right)\right)\sum_{\nu | u}\left(\frac{\nu}{7}\right) \quad \textit{for} \quad 2 \,|\, \beta,$$

$$= 0 \quad \textit{for} \quad 2 \dagger \beta.$$

Here $\left(\frac{u}{3}\right)$, $\left(\frac{u}{7}\right)$, $\left(\frac{\nu}{7}\right)$ are Legendre-Jacobi symbols.

Thus we have generalized the formulas of Kaplan and Williams [7].

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