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SOME GENERAL PROPERTIES OF GENERALIZED MÖBIUS-LISTING'S BODIES *

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Abstract. Based on analytical representation, independent components of the bulky links are estimated, which appear after one full cutting of Generalized Möbius-Listing's Bodies, with radial cross-section - regular $\boldsymbol{m}$ angular polygon and established:

1. minimal numbers of components;
2. maximal numbers of components;
3. the total number of fundamentally different variants.

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Definitions and notation

$PR_m \equiv A_0A_1\ldots A_{m-1}A'_0A'_1\ldots A'_{m-1}$ denotes an orthogonal prism, whose ends $A_0A_1\ldots A_{m-1}$ and $A'_0A'_1\ldots A'_{m=1}$ are P_m, where $P_m \equiv A_0A_1\cdots A_{m-1}$ denotes a **"Plane figure with** m**-symmetry"**, in particular P_m is a "regular polygon" and m is the number of its angles or vertices.

Definition 1. $GML_m^n\{\nu\}$- **Generalized Möbius-Listing's body** is obtained by identifying the opposite ends of the prism PR_m in such a way that:

A) For any integer $n \in Z$ and $i = 0, \ldots, m-1$ each vertex A_i coincides with $A'_{i+n} \equiv A'_{\mathrm{mod_m(i+n)}}$, and each edge A_iA_{i+1}coincides with the edge

$$A'_{i+n}A'_{i+n+1} \equiv A'_{\mathrm{mod_m(i+n)}}A'_{\mathrm{mod_m(i+n+1)}}$$

correspondingly;

B) The integer $n \in Z$ is a number of rotations of the end of the prism with respect to the axis OO' before the identification; If $n > 0$ rotations are counter-clockwise, and if $n < 0$ then rotations are clockwise;

C)Axis OO' after identifying is transformed in to a closed spatial (or plane) line ("basic line") with characteristic ν (see Remark 1 in [3]).

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According to the Definition I.

1. The axis of symmetry is required to be transformed into a closed plane or space line;

2. Rotation of the end of the prism is "semi-regular" along the middle line OO'.

So without loss of generality, we shall sometimes to use analytic representation (5) in [3], which is some particular variant of formula (7) or (8) in [2]and have the following form:

$$\begin{aligned}
X(\tau,\theta) &= \left[R + \tilde{R}\cos\left(\frac{\tilde{n}\theta}{\tilde{m}}\right) + r(\tau,\psi)\cos\left(\psi + \frac{n\theta}{m}\right)\right]\cos(\theta),\\
Y(\tau,\theta) &= \left[R + \tilde{R}\cos\left(\frac{\tilde{n}\theta}{\tilde{m}}\right) + r(\tau,\psi)\cos\left(\psi + \frac{n\theta}{m}\right)\right]\sin(\theta),\\
Z(\tau,\theta) &= \tilde{R}\sin\left(\frac{\tilde{n}\theta}{\tilde{m}}\right) + r(\tau,\psi)\sin\left(\psi + \frac{n\theta}{m}\right).
\end{aligned} \tag{1}$$

Without loss of generality in this article we suppose that the radial cross section of the Generalized Möbius-Listing's body before cutting is a circle. So we have $GML^n_m\{0\}$ and $\tilde{R} \equiv 0$ in (1).

In previous articles we have introduced a cut operation of Generalized Möbius-Listing's surfaces and bodies [2-4]. One of the most important differences between these two cases is the geometric object itself. In the case of the GML surfaces [4] of the cut line is reflected in the radial section as a point on the line, but in the case of GML bodies [2-3] - a line dissecting plane figure, in this article it is a m-symmetrical polygon! On this maybe three fundamentally different cut cases (surfaces, along which it is possible to carry out cutting operation):

Definition 2. For s-**surfaces**, without loss of generality, we will use the following notations:

1. $S_{1,j}$ - **surface** of $GML^n_m\{\nu\}$ is a slit-surface $GML^k_2\{\nu^*\}$ such that the ends of the straight line (radial cross section) are situated on the sides with the numbers 1 (or A_0A_1) and j (or $A_{j-1}A_j$ when $j = 2, 3 \cdots, \lfloor m/2 \rfloor + 1$, here $\lfloor m/2 \rfloor$ is an integer part of the fraction) correspondingly of the plane figures (m symmetric polygon) of the radial cross section of the $GML^n_m\{\nu\}$ body;

1*. SB - **surface** of the $GML^n_m\{\nu\}$ body is such $S_{1,\lfloor m/2 \rfloor+1}$ slit- surface, whose radial cross section (straight line) contains the center of symmetry and does not contain vertices of the radial cross section of the $GML^n_m\{\nu\}$ body ;

2. $VS_{0,j}$ - **surface** of $GML^n_m\{\nu\}$ body is a slit-surface $GML^k_2\{\nu^*\}$, whose radial cross section (straight line) is situated on the edges with the numbers j (where $j = 2, 3, \cdots, \lfloor (m-1)/2 \rfloor + 1$) and contains vertex number 0 of the radial cross section of the $GML^n_m\{\nu\}$ body ;

2*. $VBS_{0,\lfloor (m-1)/2 \rfloor+1}$ - **surface** of the $GML^n_m\{\nu*\}$ body is a slit-surface $GML^k_2\{\nu^*\}$, whose radial cross section (straight line) is situated on the edge with number$\lfloor (m-1)/2 \rfloor + 1$, contains the center of symmetry of polygon and vertex number 0 of the radial cross section of the $GML^n_m\{\nu\}$ body;

3. $V_{0,j}$- **surface** of the $GML_m^n\{\nu\}$ body is a slit-surface $GML_2^k\{\nu^*\}$, whose radial cross section (straight line) contain correspondingly vertex numbers 0 and j (where $j = 2, 3, \dots, \lfloor m/2 \rfloor$) of the radial cross section of the $GML_m^n\{\nu\}$ body.

Remark 1. Obviously, because of the regularity of the polygon, other possible cases can be reduced to one of the above cases. In this article the square brackets in formulas denote the integer part of the fraction.

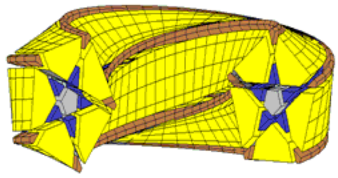
Remark 2. Here we announce the answers to the first two questions posed in the abstract.

A. If m is an even number, then for different n(more precisely, $\gcd(m.n) = 1$) - after one full cutting of GML_m^n bodies maximum $m/2+1$ independent geometric objects (components) appear (this number depends also on geometric place of the cutting line in the cross section of body), i.e. link- $(m/2+1)$ appear and only one of the components has the structure similar to a figure before cutting!

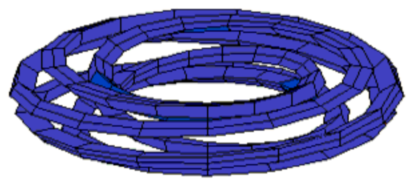
B. If m is an odd number, then for different n (more precisely $\gcd(m.n) = 1$) - after one full cutting of bodies, maximum $[m/2]+2$ independent geometric objects (components) appear (this number depends also on geometric place of the cutting line in the cross section of body), i.e. link - $([m/2]+2)$ appear and only one of the components has structure similar to a figure before cutting!

C. If m is an even number, then there exist some values of n, when after one full cutting of GML_m^n bodies only 1 independent geometric object appears (for this cutting the line should include the center of symmetry of the radial cross section of body), i.e. knot (link (1) appears, index of which is defined by $gcd(m.n)$!

D. If m is an odd number, then there exist some values of n, when after one cutting of GML_m^n -at least 2 independent geometric objects appear (for this cutting line should include the center of symmetry of the radial cross section of body), i.e. (link (2) appears), index of which is defined by $\gcd(m.n)$!



(a) The case, when after cutting of GML_5^3 four different objects sppear-link5.



(b) The case, when after cutting of GML_6^2 only one object-knot (link 1).

Figure 1

Theorem. *The total number of fundamentally different variants V_m in terms of* ***geometric shapes*** *of the elements which appear after cutting process of GML_m^n body doesn't depend on number of twisting n and:*

1. if $m = 2k+1$ is prime number then

$$V_m = V_{2k+1} = 8k+1, \tag{2}$$

2. if $m = 2k + 1$ is an odd and not a prime number, to have

$$V_m = V_{2k+1} = 8k + 1 + 3Nk + 3N + \sum_{i=1}^{N} 2\Big[\frac{k}{\gamma_i}\Big], \tag{3}$$

3. if $m = 2k$ is an even number then

$$V_m = V_{2k} = 8k - 5 + 3Nk - N + \sum_{i=1}^{N} 2\Big[\frac{(k-1)}{\gamma_i}\Big], \tag{4}$$

where N is a number of nontrivial divisors $(\gamma_1, \gamma_2, \ldots, \gamma_N)$ of number of symmetry m, i.e. $\gamma_i \neq 1$ and $\gamma_i \neq m$ for each $i = 1, 2, \ldots, N$.

Remark 3. Despite the fact that before the cut generalized Möbius-Listing's body has the zero characteristic, i.e. GML_m^n=$GML_m^n\{0\}$, after the cut appears the geometric object whose components have non zero characteristics and this number strictly depends on the number of rotation n and the number of symmetry m. This points to the fact that after the cut for different values of the rotation number n may be shapes of radial cross sections of appearing objects are similar, but differ in baseline characteristics $\nu_1 \neq \nu_2 \neq 0$ (see[3] for GML_5^n).

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