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FORMULATION OF FICTITIOUS LOAD METHOD IN POLAR COORDINATES SYSTEM FOR ELASTICITY PROBLEMS

Zirakashvili N.

Abstract. For the solution of boundary value problems and boundary-contact problems of elasticity in polar coordinates system the boundary element method, namely the fictitious load method, for domains limited with axes of system polar coordinates is formulated. The circular boundary is divided on the small size arcs and linear part is divided on the small size segments.

Keywords and phrases: Polar coordinates system, elasticity problems, boundary element method, fictitious load method.

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1. Introduction. In this paper for the solution of a boundary value problem and boundary-contact problems of elasticity in polar coordinates system the boundary element method, namely the fictitious load method, for domain limited with axes of system polar coordinates is formulated. The circular boundary there is divided not into the small size segment (as it is used in articles by various authors [1-5]), but on the small size of arcs, and linear part is divided on the small size segments. In this case the considered domain is described more precisely, than in the case when the boundary domain is divided on small segments, and, as a result, the decision turns out more exact. The numerical decision is built on the basis of the analytical decisions received beforehand for simple singular problems that satisfies approximately the given boundary conditions on each elements of a contour. The singular (fundamental) solution follows from reviewing the problem about action of the concentrated force $\overrightarrow{F}=(F_x,F_y)$ in a point of the infinite elastic medium which is known as Calvin's problem.

2. Fundamental solution in polar coordinates system. Solution of Calvin's problem, that is given in [2] in Cartesian coordinates x, y $(-\infty<x<\infty,-\infty<y<\infty)$, in polar coordinates r, ϑ $(0\le r<\infty, 0\le\vartheta<2\pi)$ will be expressed by the following function: $g_1(r,\vartheta)=-\frac{1}{4\pi(1-\nu)}\ln r$. Displacements, for example, will be written as

$$\begin{aligned}\tilde{u}_x(r,\vartheta)&=\tfrac{F_x}{2G}\left[(3-4\nu)\,g_1-r\cos\vartheta g_{1,x}\right]+\tfrac{F_y}{2G}\left(-r\sin\vartheta g_{1,x}\right),\\ \tilde{u}_y(r,\vartheta)&=\tfrac{F_x}{2G}\left(-r\cos\vartheta g_{1,y}\right)+\tfrac{F_y}{2G}\left[(3-4\nu)\,g_1-r\sin\vartheta g_{1,y}\right],\end{aligned}\tag{1}$$

and stresses will be written as

$$\begin{aligned}\tilde{\sigma}_{xx}(r,\vartheta)&=F_x\left[2\left(1-\nu\right)g_{1,x}-r\cos g_{1,xx}\right]+F_y\left(2\nu g_{1,y}-r\sin\vartheta g_{1,xx}\right),\\ \tilde{\sigma}_{yy}(r,\vartheta)&=F_x\left(2\nu g_{1,x}-r\cos\vartheta g_{1,yy}\right)+F_y\left[2\left(1-\nu\right)g_{1,y}-r\sin\vartheta g_{1,yy}\right],\\ \tilde{\sigma}_{xy}(r,\vartheta)&=F_x\left[\left(1-2\nu\right)g_{1,y}-r\cos\vartheta g_{1,xx}\right]+F_y\left[\left(1-2\nu\right)g_{1,x}-r\sin\vartheta g_{1,xy}\right],\end{aligned}\tag{2}$$

where $G = \frac{E}{2(1+\nu)}$ is shear modulus, ν is Poisson's ratio, E is Young's modulus; $g_{1,x} = -\frac{1}{4\pi(1-\nu)}\frac{\cos\vartheta}{r}$, $g_{1,y} = -\frac{1}{4\pi(1-\nu)}\frac{\sin\vartheta}{r}$, $g_{1,xx} = \frac{1}{4\pi(1-\nu)}\frac{\cos(2\vartheta)}{r^2}$, $g_{1,yy} = -\frac{1}{4\pi(1-\nu)}\frac{\cos(2\vartheta)}{r^2}$, $g_{1,xy} = \frac{1}{4\pi(1-\nu)}\frac{\cos(2\vartheta)}{r^2}$.

Using the superposition principle, we can solve problems for the infinite elastic body, which is valid in arbitrary points of the system of the concentrated forces. The continuous distribution of such forces along some curve of plane allows to consider problem in which forces are given on this curve.

3. Continuous distribution of a constant forces along curve and numerical procedure. We consider the following problem. In an infinite elastic body along the arc $\vartheta_1 \leq \vartheta \leq \vartheta_2$ of a circle with a radius r are distributed constant forces $t_r = P_r$ and $t_\vartheta = P_\vartheta$. This problem can be solved, integrating fundamental solution.

Omitting the details of the transformation, from (1) the following formula for displacements

$$\begin{aligned}
\bar{u}_x &= \tfrac{r^2}{8G\pi(1-\nu)}\left\{P_\vartheta\left[(3-4\nu)(\vartheta_1-\vartheta_2)\ln r + 0.5(\vartheta_1-\vartheta_2) + 0.5(\sin(2(\vartheta-\vartheta_1))\right.\right.\\
&\quad \left.\left.+\sin(2(\vartheta-\vartheta_2)))\right] - 0.25P_r\cos(2(\vartheta-\vartheta_1)) - \cos(2(\vartheta-\vartheta_2))\right\},\\
\bar{u}_y &= -\tfrac{r^2}{8G\pi(1-\nu)}\left\{0.25P_\vartheta(\cos(2(\vartheta-\vartheta_1)) - \cos(2(\vartheta-\vartheta_2))) + P_r\left[-(3-4\nu)\right.\right.\\
&\quad \left.\left.\times(\vartheta_1-\vartheta_2)\ln r + 0.5(\vartheta_1-\vartheta_2) + 0.25(\sin(2(\vartheta-\vartheta_1)) - \sin(2(\vartheta-\vartheta_2)))\right]\right\},
\end{aligned} \tag{3}$$

will be found, and from (2) the following formulas for stresses will be found

$$\begin{aligned}
\bar{\sigma}_{yy} &= -\tfrac{r}{8\pi(1-\nu)}\Big\{P_\vartheta\Big[(4\nu-1)(\sin(\vartheta-\vartheta_1) - \sin(\vartheta-\vartheta_2)) - \tfrac{1}{3}(\sin(3(\vartheta-\vartheta_1))\\
&\quad -\sin(3(\vartheta-\vartheta_2)))\Big] - P_r\Big[(5-4\nu)(\cos(\vartheta-\vartheta_1) - \cos(\vartheta-\vartheta_2))\\
&\quad -\tfrac{1}{3}(\cos(3(\vartheta-\vartheta_1)) - \cos(3(\vartheta-\vartheta_2)))\Big]\Big\},\\
\bar{\sigma}_{xx} &= -\tfrac{r}{8\pi(1-\nu)}\Big\{P_\vartheta\Big[(5-2\nu)(\sin(\vartheta-\vartheta_1) - \sin(\vartheta-\vartheta_2)) + \tfrac{1}{3}(\sin(3(\vartheta-\vartheta_1))\\
&\quad -\sin(3(\vartheta-\vartheta_2)))\Big] - P_r\Big[(4\nu-1)(\cos(\vartheta-\vartheta_1) - \cos(\vartheta-\vartheta_2))\\
&\quad +\tfrac{1}{3}(\cos(3(\vartheta-\vartheta_1)) - \cos(3(\vartheta-\vartheta_2)))\Big]\Big\},\\
\bar{\sigma}_{xy} &= -\tfrac{r}{8\pi(1-\nu)}\Big\{P_\vartheta\Big[(4\nu-3)(\cos(\vartheta-\vartheta_1) - \cos(\vartheta-\vartheta_2)) + \tfrac{1}{3}(\cos(3(\vartheta-\vartheta_1))\\
&\quad -\cos(3(\vartheta-\vartheta_2)))\Big] + P_r\Big[(5-4\nu)(\sin(\vartheta-\vartheta_1) - \sin(\vartheta-\vartheta_2))\\
&\quad -\tfrac{1}{3}(\sin(3(\vartheta-\vartheta_1)) - \sin(3(\vartheta-\vartheta_2)))\Big]\Big\}.
\end{aligned} \tag{4}$$

The obtained analytical solution is the basis of boundary element method for finding the numerical solution of the boundary value problem elasticity. In a particular example, we discussed the physical aspects of this method. Consider the boundary value problem for an infinite body with a cavity (in our case a circular). Consider the plane deformation. The boundary of the hole, which in our case a circle, denoted by C. Local coordinates n and s respectively directed perpendicular and tangent to curve C. Assume that the wall cavity is everywhere exposed to the same normal stress $\sigma_n = -p$ (i.e. compression) and tangential stress σ_s is zero. We want to find the displacement and stress in the body caused by the load at the boundary.

For the numerical solution of this problem we proceed as follows. We divide the boundary C with a small arcs of N amount (elements) adjacent to each other. Because these items are small, we can assume that each element along its length is exposed to normal stress $\sigma_n = -p$, but is free from the action of tangent stress. Boundary conditions in this case take the form

$$\sigma_n^i = -p, \qquad \sigma_s^i = 0, \qquad (i = 1, \ldots N)\,. \tag{5}$$

For each boundary element we choose concentrate forces uniformly distributed throughout its length. For example, for the j-th element we assume a continuous distribution of tangential P_s^j and normal P_n^j stresses. Also, for the j-th element we have fictitious stresses P_s^j and P_n^j and also real stresses σ_s^j and σ_n^j induced by the stresses applied to all boundary elements.

Using (3) and (4), we can calculate the real stresses σ_s^j and σ_n^j at the midpoints of all elements. Thus we obtain the formulas

$$\sigma_s^i = \sum_{j=1}^{N} A_{ss}^{ij} P_s^j + \sum_{j=1}^{N} A_{sn}^{ij} P_n^j, \qquad \sigma_n^i = \sum_{j=1}^{N} A_{ns}^{ij} P_s^j + \sum_{j=1}^{N} A_{nn}^{ij} P_n^j, \qquad i = 1, \ldots, N, \tag{6}$$

where A_{ss}^{ij}, A_{sn}^{ij}, A_{ns}^{ij}, A_{nn}^{ij} are the boundary coefficients for the influence of stresses for the problem under consideration. For example, the coefficient A_{ns}^{ij} gives the real normal stress at the middle of the i-th arc (σ_n^i) induced by the constant unit tangential load $(P_s^j = 1)$ applied to the j-th arc.

From (5) and (6) we receive the following system of $2N$ linear equations with $2N$ unknowns (P_n^j and P_s^j, $j = 1, \ldots, N$)

$$0 = \sum_{j=1}^{N} A_{ss}^{ij} P_s^j + \sum_{j=1}^{N} A_{sn}^{ij} P_n^j, \quad -p = \sum_{j=1}^{N} A_{ns}^{ij} P_s^j + \sum_{j=1}^{N} A_{nn}^{ij} P_n^j, \quad i = 1, \ldots, N. \tag{7}$$

After solving system (7) by any numerical method, we can calculate the displacements and stresses at any point of the body.

4. Boundary coefficient influence. We will write expressions for normal and tangential displacements and stresses at midpoint i-th element due to the action of fictitious loads P_n^j and P_s^j, $(j = 1, \ldots N)$, applied to the j-th element. The boundary coefficients for displacements are determined from the formulas

$$\begin{aligned}
u_s^i = {} & P_s^j \left\{ \tfrac{\bar{r}^2}{8G\pi(1-\nu)} \left[(3-4\nu)\left(\vartheta_1 - \vartheta_2\right) \ln \bar{r} \right.\right. \\
& \left.\left. +0.5\left(\vartheta_1 - \vartheta_2\right) + 0.25 \sin\left(2\left(\bar{\vartheta} - \vartheta_1\right)\right) - \sin\left(2\left(\bar{\vartheta} - \vartheta_2\right)\right)\right]\right\} \\
& + P_n^j \left\{ \tfrac{-\bar{r}^2}{32G\pi(1-\nu)} \left[\cos\left(2\left(\bar{\vartheta} - \vartheta_1\right)\right) - \cos\left(2\left(\bar{\vartheta} - \vartheta_2\right)\right)\right]\right\}, \\
u_n^i = {} & P_s^j \left\{ -\tfrac{\bar{r}^2}{32G\pi(1-\nu)} \left[\cos\left(2\left(\bar{\vartheta} - \vartheta_1\right)\right) - \cos\left(2\left(\bar{\vartheta} - \vartheta_2\right)\right)\right]\right\} \\
& + P_n^j \left\{ \tfrac{\bar{r}^2}{8G\pi(1-\nu)} \left[(3-4\nu)\left(\vartheta_1 - \vartheta_2\right) \ln \bar{r} - 0.5\left(\vartheta_1 - \vartheta_2\right) \right.\right. \\
& \left.\left. -0.25\left(\sin\left(2\left(\bar{\vartheta} - \vartheta_1\right)\right) - \sin\left(2\left(\bar{\vartheta} - \vartheta_2\right)\right)\right)\right]\right\},
\end{aligned} \tag{8}$$

and for stresses are determined from the formulas

$$
\begin{aligned}
\sigma_n^i = {} & P_s^j \left\{ \tfrac{-\bar{r}}{8\pi(1-\nu)} \left[(4\nu-1) \left(\sin\left(\bar{\vartheta}-\vartheta_1\right) - \sin\left(\bar{\vartheta}-\vartheta_2\right) \right) - \tfrac{1}{3} \left(\sin\left(3\left(\bar{\vartheta}-\vartheta_1\right)\right) \right.\right.\right. \\
& \left.\left.\left. - \sin\left(3\left(\bar{\vartheta}-\vartheta_2\right)\right)\right)\right]\right\} + P_n^j \left\{ \tfrac{\bar{r}}{8\pi(1-\nu)} \left[(5-4\nu) \left(\cos\left(\bar{\vartheta}-\vartheta_1\right) - \cos\left(\bar{\vartheta}-\vartheta_2\right) \right) \right.\right. \\
& \left.\left. - \tfrac{1}{3} \left(\cos\left(3\left(\bar{\vartheta}-\vartheta_1\right)\right) - \cos\left(3\left(\bar{\vartheta}-\vartheta_2\right)\right)\right)\right]\right\}, \\
\sigma_s^i = {} & P_s^j \left\{ \tfrac{-\bar{r}}{8\pi(1-\nu)} \left[(4\nu-3) \left(\cos\left(\bar{\vartheta}-\vartheta_1\right) - \cos\left(\bar{\vartheta}-\vartheta_2\right) \right) + \tfrac{1}{3} \left(\cos\left(3\left(\bar{\vartheta}-\vartheta_1\right)\right) \right.\right.\right. \\
& \left.\left.\left. - cos\left(3\left(\bar{\vartheta}-\vartheta_2\right)\right)\right)\right]\right\} + P_n^j \left\{ \tfrac{-\bar{r}}{8\pi(1-\nu)} \left[(3-4\nu) \left(\sin\left(\bar{\vartheta}-\vartheta_1\right) - \sin\left(\bar{\vartheta}-\vartheta_2\right) \right) \right.\right. \\
& \left.\left. - \tfrac{1}{3} \left(\sin\left(3\left(\bar{\vartheta}-\vartheta_1\right)\right) - \sin\left(3\left(\bar{\vartheta}-\vartheta_2\right)\right)\right)\right]\right\}, \\
\sigma_t^i = {} & P_s^j \left\{ \tfrac{-\bar{r}}{8\pi(1-\nu)} \left[(5-2\nu) \left(\sin\left(\bar{\vartheta}-\vartheta_1\right) - \sin\left(\bar{\vartheta}-\vartheta_2\right) \right) + \tfrac{1}{3} \left(\sin\left(3\left(\bar{\vartheta}-\vartheta_1\right)\right) \right.\right.\right. \\
& \left.\left.\left. - \sin\left(3\left(\bar{\vartheta}-\vartheta_2\right)\right)\right)\right]\right\} + P_n^j \left\{ \tfrac{\bar{r}}{8\pi(1-\nu)} \left[(4\nu-1) \left(\cos\left(\bar{\vartheta}-\vartheta_1\right) - \cos\left(\bar{\vartheta}-\vartheta_2\right) \right) \right.\right. \\
& \left.\left. + \tfrac{1}{3} \left(\cos\left(3\left(\bar{\vartheta}-\vartheta_1\right)\right) - \cos\left(3\left(\bar{\vartheta}-\vartheta_2\right)\right)\right)\right]\right\},
\end{aligned}
\tag{9}
$$

where $\bar{r}$ and $\bar{\vartheta}$ are the coordinates in the local coordinate system with center at midpoint i-th element. Displacements and stresses in the i-th element in the general case are functions of the components P_s^j and P_n^j $(j = 1, \ldots N)$ fictitious loads in all N elements. Thus, according to (8) and (9) we can write

$$
u_s^i = \sum_{j=1}^{N} B_{ss}^{ij} P_s^j + \sum_{j=1}^{N} B_{sn}^{ij} P_n^j, \quad u_n^i = \sum_{j=1}^{N} B_{ns}^{ij} P_s^j + \sum_{j=1}^{N} B_{nn}^{ij} P_n^j,
$$
$$
\sigma_s^i = \sum_{j=1}^{N} A_{ss}^{ij} P_s^j + \sum_{j=1}^{N} A_{sn}^{ij} P_n^j, \quad \sigma_n^i = \sum_{j=1}^{N} A_{ns}^{ij} P_s^j + \sum_{j=1}^{N} A_{nn}^{ij} P_n^j.
$$

The boundary influence coefficients $B_{ss}^{ij}, \ldots$ and $A_{ss}^{ij}, \ldots$ in these equations are given expression in curly brackets in (8) and (9).

By using the above formulated method numerical results of some problems of elasticity are obtained. Those results are not given in this article and we can't present them here.

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Author's address:

Natela Zirakashvili
I. Vekua Institute of Applied Mathematics of
Ivane Javakhishvili Tbilisi State University
2, University St., Tbilisi 0186
Georgia
E-mail: natzira@yahoo.com