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ON THE NUMBER OF REPRESENTATIONS OF POSITIVE INTEGERS BY CERTAIN BINARY QUADRATIC FORMS

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Abstract. We shall obtain the exact formulas for the number of representations by all primitive binary quadratic forms with discriminants -76 and -80.

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Let $r(n;f)$ denote a number of representations of a positive integer n by a positive definite quadratic form f with a number of variables s. It is well known that, for case $s>4$, $r(n;f)$ can be represented as $r(n;f)=\rho(n;f)+\nu(n;f)$, where $\rho(n;f)$ is a "singular series" and $\nu(n;f)$ is a Fourier coefficient of cusp form. This can be expressed in terms of the theory of modular forms by stating that

$$\vartheta(\tau;f)=E(\tau;f)+X(\tau),\ \ \vartheta(\tau;f)=1+\sum_{n=1}^{\infty}r(n;f)\,Q^n, \tag{1}$$

where $\tau\in H=\{\tau:\operatorname{Im}\tau>0\}$, $Q=e^{2\pi i\tau}$, $X(\tau)$ is a cusp form and $E(\tau;f)=1+\sum_{n=1}^{\infty}\rho(n;f)\,Q^n$, is the Eisenstein series corresponding to f.

Siegel [1] proved that if the number of variables of a quadratic form f is $s>4$, then

$$E(\tau;f)=F(\tau;f), \tag{2}$$

where $F(\tau;f)$ denotes a theta-series of a genus containing a primitive quadratic form f (both positive-definite and indefinite). From formula (2) follows the well-known Siegel's theorem [1]: The sum of the singular series corresponding to the quadratic form f is equal to the average number of representations of a natural number by a genus that contains the form f.

Later Ramanathan [2] proved that for any primitive integral quadratic form with $s\geq 3$ variables (except for zero forms with variables $s=3$ and zero forms with variables $s=4$ whose discriminant is a perfect square), there is a function

$$E(\tau,z;f)=1+\frac{e^{\frac{(2m-s)\pi i}{4}}}{|d|^{\frac{1}{2}}}\sum_{q=1}^{\infty}\sum_{\substack{H=-\infty\\(H,q)=1}}^{\infty}\frac{S(fH;q)}{q^{\frac{s}{2}}(q\tau-H)^{\frac{m}{2}}(q\overline{\tau}-H)^{\frac{s-m}{2}}|q\tau-H|^{z}}\,,$$

which he called the Eisenstein-Siegel series and which is regular for any fixed τ when $\operatorname{Im}\tau>0$ and $\operatorname{Re}z>2-\frac{s}{2}$, analytically extendable in a neighborhood of $z=0$, and that

$$F(\tau;f)=E(\tau,z;f)\big|_{z=0}. \tag{3}$$

Here m and d are respectively the inertia index and discriminant of f, $S(fH,q)$ is the Gaussian sum. For $s>4$ the function $E(\tau,z;f)\big|_{z=0}$ coincides with the function $E(\tau;f)$ and the formula (3) with Siegel's formula (2).

In [3] we proved that the function $E(\tau,z;t)$ is analytically extendable in the neighborhood of $z=0$ also in the case where f is any nonzero integral binary quadratic form and that $F(\tau;f)=\frac{1}{2}\,E(\tau,z;f)\big|_{z=0}$. Further, having defined the Eisenstein series $E(\tau;f)$ by the formulas $E(\tau;f)=\frac{1}{2}\,E(\tau,z;f)\big|_{z=0}$ for $s=2$ and $E(\tau;f)=E(\tau,z;f)\big|_{z=0}$ for $s>2$, we have $E(\tau;f)=1+\frac{1}{2}\sum\limits_{n=12}^{\infty}\rho(n;f)Q^n$ for $s=2$ and $E(\tau;f)=1+\sum\limits_{n=1}^{\infty}\rho(n;f)Q^n$ for $s>2$. Moreover, convenient formulas are obtained for calculating values of the function $\rho(n;f)$ in the case where f is any positive integral form of variables $s\geq 2$ (see [4], [5], [6]).

Thus, if the genus of the quadratic form f contains one class, then according to Siegel's theorem, $r(n;f)=\rho(n;f)$ for $s\geq 3$, $r(n;f)=\frac{1}{2}\,\rho(n;f)$ for $s=2$ and in that case the problem for obtaining "exact" formulas for $r(n;f)$ is solved completely.

In this paper we will obtain exact formulas for the number of representations of a natural number by all primitive binary quadratic forms with discriminants -76 and -80 belonging to multi-class genera. It is well known that there is exactly one reduced form in each equivalence class of positive definite binary quadratic forms.

The reduced forms with discriminant -76 are the Gaussian forms $f_1=x_1^2+19\,x_2^2$, $f_2=4\,x_1^2+2\,x_1\,x_2+5\,x_2^2$, $f_3=4\,x_1^2-2\,x_1\,x_2+5\,x_2^2$, which belong to the same genera.

The set of forms with discriminant -80 splits into two genera of forms and each genus consists of two reduced classes of forms which are respectively

$$f_4=x_1^2+20\,x_2^2,\quad f_5=4\,x_1^2+5\,x_2^2\ \text{ and }\ f_6=3\,x_1^2+2\,x_1\,x_2+7\,x_2^2,\quad f_7=3\,x_1^2-2\,x_1\,x_2+7\,x_2^2.$$

The exact formulas for $r(n;f)$ in case of the binary forms f_1, f_2 and f_3 are obtained in [7]. The proof for odd number n is based on Thirihlet's theorem [7]. In the same work [7] in case of forms f_4 and f_5 it wasn't succeeded to apply this theorem and formulae only for even n were received.

Lomadze [8], [9] considered the forms f_1, f_4 and f_5. He obtained the formulae with remainder members being the coefficients of certain cusp forms without arithmetical meaning. All formulas which are obtained in this paper have no remainder member or one remainder member and its arithmetical meaning is shown.

It is obvious that $r(n;f_6)=r(n;f_7)$. Thus, by Siegel's theorem $r(n;f_6)=r(n;f_7)=\frac{1}{2}\,\rho(n;f_6)$. The function $\rho(n;f_6)$ may be calculated by the paper [4]. So we have

Theorem 1. *Let $n=2^\alpha\,5^\beta u$, $(u,10)=1$. Then*

$$\begin{aligned} r(n;f_6)=r(n;f_7)&=\frac{1}{2}\Big(1-\Big(\frac{u}{5}\Big)\Big)\sum_{\nu|u}\Big(\frac{-5}{\nu}\Big)\ \textit{for}\ \alpha=0,\quad u\equiv 3\,(\mathrm{mod}\,4),\\ &=\Big(1-(-1)^\alpha\Big(\frac{u}{5}\Big)\Big)\sum_{\nu|u}\Big(\frac{-5}{\nu}\Big)\ \textit{for}\ 2|\alpha,\quad \alpha\geq 2,\quad u\equiv 3\,(\mathrm{mod}\,4),\\ &\qquad\qquad\qquad\qquad \textit{or for}\quad \alpha\geq 2,\quad 2\dagger\alpha,\quad u\equiv 1\,(\mathrm{mod}\,4),\\ &=0\ \ \textit{otherwise}. \end{aligned}$$

In order to use the theory of modular forms in case of the binary forms f_k ($k = 1,2,3,4,5$) it is necessary to construct the cusp form $X(\tau)$ which is so-called remainder member in the formula (1). For this purpose we use the modular properties of the generalized theta-function defined in [6] as follows:

$$\vartheta_{gh}(\tau; p_\nu, f) = \sum_{x \equiv g \,(\mathrm{mod}\, N)} (-1)^{\frac{h' A(x-g)}{N^2}} p_\nu(x)\, e^{\frac{\pi i \tau x' A x}{N^2}}.$$

Here A is an integral matrix of f, $x \in \mathbb{Z}^{\sim}$, g and h are the special vectors with respect to the form f, $p_\nu(x)$ is a spherical function of the ν-th order corresponding to f; N is a step of the form f. In particular, if g and h are zero vectors and $p_0(x) = 1$, then $\vartheta_{gh}(\tau; p_0, f) = \vartheta(\tau; f)$. We assume, that $\vartheta_{gh}(\tau; p_0, f) = \vartheta_{gh}(\tau; f)$, where $p_0 = 1$.

By means of the theory of modular forms we prove the following two theorems:

Theorem 2. *Let* $g = \begin{pmatrix} 38 \\ 0 \end{pmatrix}$, $h = \begin{pmatrix} 38 \\ 0 \end{pmatrix}$. *Then we have*

$$\begin{aligned} \vartheta(\tau; f_1) &= \frac{1}{2}\, E(\tau; f_1) + \frac{2}{3}\, \vartheta_{gh}(\tau; f_2), \\ \vartheta(\tau; f_2) &= \vartheta(\tau; f_3) = \frac{1}{2}\, E(\tau; f_1) - \frac{1}{3}\, \vartheta_{gh}(\tau; f_2). \end{aligned}$$

Theorem 3. *Let* $g = \begin{pmatrix} 4 \\ 12 \end{pmatrix}$, $h = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$. $f = 10\, x_1^2 + 10\, x_2^2$. *Then we have*

$$\begin{aligned} \vartheta(\tau; f_4) &= \frac{1}{2}\, E(\tau; f_4) + \vartheta_{gh}(\tau; f), \\ \vartheta(\tau; f_5) &= \frac{1}{2}\, E(\tau; f_4) - \vartheta_{gh}(\tau; f). \end{aligned}$$

From Theorems 2 and 3 we obtain respectively Theorems 4 and 5.

Theorem 4. *Let* $n = 2^\alpha\, 19^\beta u$, $(u, 38) = 1$. *Then*

$$\begin{aligned} r(n; f_k) &= \frac{1}{3}\left(1 + \left(\frac{u}{19}\right)\right) \sum_{\nu | u} \left(\frac{-19}{\nu}\right) + \nu(n; f_k) \;\; \textit{if} \;\; \alpha = 0, \\ &= \left(1 + \left(\frac{u}{19}\right)\right) \sum_{\nu | u} \left(\frac{-19}{\nu}\right) \;\; \textit{if} \;\; 2|\alpha, \;\; \alpha > 0, \\ &= 0 \;\; \textit{if} \;\; 2 \dagger \alpha, \quad \textit{where} \;\; k = 1,2,3 \quad \textit{and} \end{aligned}$$

$$\nu(n; f_1) = \frac{2}{3} \sum_{\substack{n = x_1^2 + x_1 x_2 + 5x_2^2 \\ 2 \dagger x_1}} (-1)^{x_2}, \quad \nu(n; f_2) = \nu(n; f_3) = -\frac{1}{3} \sum_{\substack{n = x_1^2 + x_1 x_2 + 5x_2^2 \\ 2 \dagger x_1}} (-1)^{x_2}.$$

Theorem 5. *Let $n = 2^\alpha\, 5^\beta u$, $(u, 10) = 1$. Then*

$$
\begin{aligned}
r(n; f_k) &= \frac{1}{2}\Big(1 + \Big(\frac{u}{5}\Big)\Big)\sum_{\nu|u}\Big(\frac{-5}{\nu}\Big) + \nu(n; f_k) \;\; \textit{if} \;\; \alpha = 0, \;\; u \equiv 1 \pmod 4,\\
&= \Big(1 + \Big(\frac{u}{5}\Big)\Big)\sum_{\nu|u}\Big(\frac{-5}{\nu}\Big) \;\; \textit{of} \;\; 2|\alpha, \;\; \alpha > 0, \;\; u \equiv 1 \pmod 4,\\
&= \Big(1 - \Big(\frac{u}{5}\Big)\Big)\sum_{\nu|u}\Big(\frac{-5}{\nu}\Big) \;\; \textit{if} \;\; 2\dagger\alpha, \;\; \alpha > 1, \;\; u \equiv 3 \pmod 4,\\
&= 0 \;\; \textit{otherwise},
\end{aligned}
$$

where $k = 4, 5$ and

$$
\nu(n; f_4) = \sum_{10n=(10x_1+1)^2+(10x_2+3)^2} (-1)^{x_1+x_2}, \quad \nu(n; f_5) = -\sum_{10n=(10x_1+1)^2+(10x_2+3)^2} (-1)^{x_1+x_2}.
$$

REFERENCES

1. Siegel C. L. Über die analytische Theorie der quadratischen Formen. *Ann. of Math.* (2) **36**, 3 (1935), 527-606.
2. Ramanathan K. G. On the analytic theory of quadratic forms. *Acta Arith.* **21** (1972), 423-436.
3. Vepkhvadze T. V. On a formula of Siegel (Russian). *Acta Arith.* **40**, 2 (1981/82), 125-142.
4. Vepkhvadze T. V. The representation of numbers by positive Gaussian binary quadratic forms (Russian), *Trudy Tbiliss. Mat. Inst. Razmadze* **40** (1971), 21-587.
5. Vepkhvadze T. V. The representation of numbers by positive binary quadratic forms of odd discriminant (Russian), *Trudy Tbiliss. Mat. Inst. Razmadze* **45** (1974), 4-40.
6. Vepkhvadze T. Modular properties of theta-functions and representation of numbers by positive quadratic forms. *Georgian Math. J.* **4**, 4 (1997), 385-400.
7. Kaplan P. and Williams K. S. On the number of representations of a positive integer by a binary quadratic form. *Acta Arith.* **114**, 1 (2004), 87-98.
8. Lomadze G. A. The representation of numbers by certain binary quadratic forms (Russian). *Izv. Vyssh. Uchebn. Zaved. Matematika* **11**, 11 (102) (1970), 71–75.
9. Lomadze G. A. On the representation of numbers by positive binary diagonal quadratic forms (Russian). *Mat. Sb.* (N.S.) **68**, 110 (1965), 282-319.

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