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SOME EXACT SOLUTIONS OF THE NONLINEAR 2D BURGER'S EQUATION

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Abstract. Soliton-like solutions of the 2D nonlinear Burger's equation are obtained. Revision of the previously received solutions is carried out.

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1. Introduction. Dynamics of shock waves, i.e. motion of thin layers - fronts separating regions with different densities and temperatures is described by Burger's equation. The thickness of such front of shock waves is finite and isolated in space, that is why they call it soliton-like structure.

In the given paper we study the soliton-like solutions of the following 2D nonlinear Burger's equation [1]

$$(u_t + uu_x - u_{xx})_x + u_{yy} = 0. \tag{1}$$

Note that being described by Eq. (1) shock waves are weakly two-dimensional in the sense that the scale of wave-length variation along the y-axis is much more than along the x-axis.

The aim of the paper is to elucidate the process of obtaining of some exact solutions of Eq. (1) and the revision of the appropriate solutions had been got in [2].

2. Exact solutions. At the end of the last century straight methods of finding of soliton-like solutions of nonlinear evolution equations were being activated [2 - 6]. According to this method [7] we suppose that Eq. (1) has the solution of the following form

$$u(x,y,t) = A\partial_x w\left[z(x,y,t)\right] + B. \tag{2}$$

Substitution of Eq. (2) into Eq. (1) yields

$$\begin{aligned}
&w^{(3)} z_y^2 z_x + w'' z_{yy} z_x + w^{(3)} z_t z_x^2 + Bw^{(3)} z_x^3 + A\left(w''\right)^2 z_x^4 + Aw' w^{(3)} z_x^4 \\
&-w^{(4)} z_x^4 + 2w'' z_x z_{xt} + 2w'' z_y z_{xy} + w' z_{xyy} + w'' z_t z_{xx} + 3Bw'' z_x z_{xx} \\
&+5Aw' w'' z_x^2 z_{xx} - 6w^{(3)} z_x^2 z_{xx} + A\left(w'\right)^2 z_{xx}^2 - 3w'' z_{xx}^2 + w' z_{xxt} \\
&+Bw' z_{xxx} + A\left(w'\right)^2 z_x z_{xxx} - 4w'' z_x z_{xxx} - w' z_{xxxx} = 0.
\end{aligned} \tag{3}$$

Equating the coefficients at z_x^4 to zero we get the following ordinary differential equation $A\left(w'w''\right)' - w^{(4)} = 0$, having the solution

$$w(z) = -\frac{2}{A}\ln z. \tag{4}$$

Further we suppose that the function $z(x,y,t)$ has the linear with respect to x the following form

$$z(x,y,t) = P(y,t) + \exp\left[Q(y,t)x + R(y,t)\right], \tag{5}$$

where $P(y,t)$, $Q(y,t)$ and $R(y,t)$ are differentiable functions with respect to y and t variables. Let us substitute Eqs. (2), (4) and (5) into the initial Eq. (1), we get

$$\begin{aligned}
-e^{2L}Q_{yy} + e^{L}\big[&BPQ^3 - PQ^4 - Q^2P_t - 2PQQ_t + xPQ^2Q_t + PQ^2R_t + 2P_yQ_y \\
&-2xQP_yQ_y - 2xPQ_y^2 + x^2PQQ_y^2 - 2QP_yR_y - 2PQ_yR_y \\
&+2xPQQ_yR_y + PQR_y^2 + QP_{yy} - 2PQ_{yy} - xPQQ_{yy} - PQR_{yy}\big] \\
&+\big[-BP^2Q^3 + P^2Q^4 + PQ^2P_t - 2P^2QQ_t - xP^2Q^2Q_t - P^2Q^2R_t \\
&-2QP_y^2 + 2PP_yQ_y + 2xPQP_yQ_y - 2xP^2Q_y^2 - x^2P^2QQ_y^2 \\
&+2PQP_yR_y - 2P^2Q_yR_y - 2xP^2QQ_yR_y - P^2QR_y^2 + PQP_{yy} \\
&-P^2Q_{yy} - xP^2QQ_{yy} - P^2QR_{yy}\big] = 0.
\end{aligned} \tag{6}$$

Here $L = L(x,y,t) = Q(y,t)x + R(y,t)$.

If we equalize to zero the coefficients at e^{2L}, x^2e^L, xe^L, e^L, x^2, x and x^2 we get the following system

$$\begin{cases}
Q_{yy} = Q_y = Q_t = 0, \\
BPQ^2 - PQ^3 - QP_t + PQR_t - 2P_yR_y + PR_y^2 + P_{yy} - PR_{yy} = 0, \\
BP^2Q^2 - P^2Q^3 - PQP_t + P^2QR_t + 2P_y^2 - 2PP_yR_y + P^2R_y^2 - PP_{yy} + P^2R_{yy} = 0.
\end{cases} \tag{7}$$

Thus under the conditions (7) Eq. (1) has the following soliton-like solution

$$u(x,y,t) = -\frac{2Qe^L}{P+e^L} + B = -Q(y,t)\left[1 + \tanh\frac{Q(y,t)x + R(y,t) - \ln P(y,t)}{2}\right] + B. \tag{8}$$

From the first equation of (7) it follows that $Q(y,t) = k = const \neq 0$, and for the rest we get

$$\begin{cases}
k^2BP - k^3P - kP_t + kPR_t - 2P_yR_y + PR_y^2 + P_{yy} - PR_{yy} = 0, \\
k^2BP^2 - k^3P^2 - kPP_t + kP^2R_t + 2P_y^2 - 2PP_yR_y + P^2R_y^2 - PP_{yy} + P^2R_{yy} = 0.
\end{cases} \tag{9}$$

At last according to the given formalism soliton-like solution of the nonlinear Eq. (1) becomes

$$u(x,y,t) = -k\left[1 + \tanh\frac{kx + R(y,t) - \ln P(y,t)}{2}\right] + B. \tag{10}$$

In this way substituting the solutions of (9) in Eq. (10) we will get new solutions.

For example, let us consider the following cases:

I. Suppose $P(y,t) = 1$, then the system (9) is

$$\begin{cases}
R_{yy} + R_y^2 + kR_t + Bk^2 - k^3 = 0, \\
R_{yy} - R_y^2 - kR_t - Bk^2 + k^3 = 0.
\end{cases} \tag{11}$$

As it is seen, the compatibility condition for the system (11) is $R_{yy}(y,t) = 0$ or the function R should have the form

$$R(y,t) = yf_1(t) + c_1y + f_2(t) + c_2, \tag{12}$$

where $f_1(t)$ and $f_2(t)$ are arbitrary functions, but c_1 and c_2 are arbitrary constants. From system (11) we can define the constant B as

$$B = \frac{k^3 - k\left[yf_1'(t) + f_2'(t)\right] - \left[f_1(t) + c_1\right]^2}{k^2}. \tag{13}$$

To keep B constant it is necessary to have $f_1'(t) = 0$ and $f_2'(t) = b$. Thus we can define $f_1(t) = c_3$, and $f_2(t) = bt$. Then Eq. (12) becomes

$$R(y,t) = ay + bt + c, \quad (a = c_1 + c_3). \tag{14}$$

Thus the solution (10) of Eq. (1) in the case under consideration is

$$u(x,y,t) = -k\tanh\left(\frac{kx + ay + bt + c}{2}\right) - \frac{b}{k} - \frac{a^2}{k^2}. \tag{15}$$

This solution represents the solitary wave and coincides with Eq. (2.13) obtained in [2].

In case when $f_1(t) = c_1 = 0$, and $f_2(t) = k(k-B)t$ we cannot define B by Eq. (13) and the solution (10) is

$$u(x,y,t) = -k\left[1 + \tanh\frac{kx + k(k-B)t + c}{2}\right] + B. \tag{16}$$

This expression represents independent from y-variable solution and coincides with the solution (2.14) obtained in [2]. As to the solution (2.15) given in [2] is wrong because it doesn't satisfy the condition $R_{yy}(y,t) = 0$.

II. Suppose $P(y,t) = t$, then system (9) reduces to

$$\begin{cases} \left(R_{yy} + R_y^2 + kR_t\right)t + \left(Bk^2 - k^3\right)t - k = 0, \\ \left(R_{yy} - R_y^2 - kR_t\right)t - \left(Bk^2 - k^3\right)t + k = 0. \end{cases} \tag{17}$$

The system compatibility condition again is $R_{yy}(y,t) = 0$ and the function $R(y,t)$ again has the form (12). From system (17) we find

$$B = \frac{k - R_y^2 t - kR_t t + k^3 t}{k^2 t} = \frac{k - \left[f_1(t) + c_1\right]^2 t - k\left[yf_1'(t) + f_2'(t)\right] + k^3 t}{k^2 t}. \tag{18}$$

It is seen that to keep B constant the conditions $f_1'(t) = 0$ and $f_2'(t) = 1/t$ should be satisfied. Thus the appropriate functions can be defined as follows: $f_1(t) = c_3$, and $f_2(t) = \ln t$. Consequently $R(y,t) = ay + \ln t + c$, where $a = c_3 + c_1$. Then, according to (18), we can define the constant $B = k - \dfrac{a^2}{k^2}$. Thus solution (10) of Eq. (1) is

$$u(x,y,t) = -\frac{a^2}{k^2} - k\tanh\left(\frac{kx + ay + c}{2}\right), \tag{19}$$

which represents the stationary solution and generalizes solution (2.17) of [2]. In case when $f_1(t) = c_1 = 0$ and $f_2(t) = k(k-B)t + \ln t$ it becomes impossible to define B and (10) can be written as independent of y-variable solution (16).

III. When $P(y,t) = y$, then system (9) gives

$$\begin{cases} y^2\left(R_{yy} + R_y^2 + kR_t\right) - 2yR_y + y^2\left(Bk^2 - k^3\right) + 2 = 0, \\ y\left(R_{yy} - R_y^2 - kR_t\right) + 2R_y - y\left(Bk^2 - k^3\right) = 0. \end{cases} \tag{20}$$

The compatibility condition of this system is

$$\begin{cases} R_{yy} = -\dfrac{1}{y^2} \,, \\ Bk^2 - k^3 = -\dfrac{1}{y^2} - R_y^2 - kR_t + \dfrac{2}{y} R_y \,. \end{cases} \tag{21}$$

From the first condition we can define the following general form

$$R(y,t) = \ln y + y f_1(t) + c_1 y + f_2(t) + c_2, \tag{22}$$

where $f_1(t)$ and $f_2(t)$ are arbitrary functions and c_1 and c_2 are arbitrary constants. From the second equation of (21) we see that the following conditions $R_y = 1/y$ and $R_t = const$ are needed to keep the left-hand side of Eq. (21) as constant. Thus the function (22) should have the form $R(y,t) = \ln y + at + c$, which can be achieved if in Eq. (22) we choose $f_1(t) = c_1 = 0$, and $f_2(t) = at$. In addition, from the second equation of (21) we find $a = -Bk + k^2$. Thus Eq. (1) has the following solution

$$u(x,y,t) = -k\left[1 + \tanh\frac{kx + k(k-B)t + c}{2}\right] + B, \tag{23}$$

which coincides with the solution (16).

If we don't specify the form of a, then from Eq. (21) we find $B = k - \dfrac{a}{k}$, and the solution of Eq. (1) is

$$u(x,y,t) = -\frac{a}{k} - k\tanh\frac{kx + at + c}{2}. \tag{24}$$

As to the solution (2.19) of [2] it is wrong.

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