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ON THE SPACE OF SPHERICAL POLYNOMIAL WITH QUADRATIC FORMS OF FIVE VARIABLES

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Abstract. The spherical polynomials of order ν with respect to quadratic form of five variables are constructed and the basis of the spaces of these spherical polynomials is established. The space of generalized theta-series is considered.

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Let

$$Q(X) = Q(x_1, \cdots, x_r) = \sum_{1\leq i\leq j\leq r} b_{ij}x_ix_j$$

be an integer positive definite quadratic form in an even number r of variables. To $Q(X)$ we associate the even integral symmetric $r \times r$ matrix A defined by $a_{ii} = 2b_{ii}$ and $a_{ij} = a_{ji} = b_{ij}$, where $i < j$. If $X = [x_1, \cdots, x_r]^T$ denotes a column vector, X^T is its transposition, then we have $Q(X) = \frac{1}{2}X^TAX$. Let A_{ij} denote the cofactor to the element a_{ij} in $D = \det A$ and a^*_{ij} the corresponding element of A^{-1}.

A homogeneous polynomial $P(X) = P(x_1, \cdots, x_r)$ of degree ν with complex coefficients, satisfying the condition

$$\sum_{1\leq i,j\leq r} a^*_{ij}\Big(\frac{\partial^2 P}{\partial x_i\partial x_j}\Big) = 0 \qquad (1)$$

is called a spherical polynomial of order ν with respect to $Q(X)$ (see [1]), and

$$\vartheta(\tau, P, Q) = \sum_{n\in\mathbb{Z}^r} P(n)z^{Q(n)}, \qquad z = e^{2\pi i\tau}, \qquad \tau \in \mathbb{C}, \qquad \operatorname{Im}\tau > 0$$

is the corresponding generalized r-fold theta-series.

Let $P(\nu, Q)$ denote the vector space over $\mathbb{C}$ of spherical polynomials $P(X)$ of even order ν with respect to $Q(X)$. Hecke [2] calculated the dimension of the space $P(\nu, Q)$ and showed that

$$\dim P(\nu, Q) = \binom{\nu + r - 1}{r - 1} - \binom{\nu + r - 3}{r - 1}$$

and form the basis of the space of spherical polynomials of second order with respect to $Q(X)$.

For $\nu = 4$, Lomadze [3] constructed the basis of the space of spherical polynomials of the fourth order with respect to $Q(X)$.

Our goal is to construct a basis of the space $P(\nu, Q)$ with a simpler way.

Let

$$P(X) = P(x_1, x_2, x_3, x_4, x_5) = \sum_{k=0}^{\nu}\sum_{i=0}^{k}\sum_{j=0}^{i}\sum_{l=0}^{j} a_{kijl} x_1^{\nu-k} x_2^{k-i} x_3^{i-j} x_4^{j-l} x_5^{l}$$

be a spherical function of order ν with respect to the positive quadratic form $Q(x_1, x_2, x_3, x_4, x_5)$ of five variables. Hence, according to condition (1) of spherical function and considering all $\frac{\partial^2 P}{\partial x_i \partial x_j}$, we obtain

$$\begin{aligned}
&A_{11}(\nu-k+1)(\nu-k)a_{k-1,i,j,l} + 2A_{12}(\nu-k)(k-i)a_{k,i,j,l}\\
&+2A_{13}(\nu-k)(i-j+1)a_{k,i+1,j,l} + 2A_{14}(\nu-k)(j-l+1)a_{k,i+1,j+1,l}\\
&+2A_{15}(\nu-k)(l+1)a_{k,i+1,j+1,l+1} + A_{22}(k-i)(k-i+1)a_{k+1,i,j,l}\\
&+2A_{23}(k-i)(i-j+1)a_{k+1,i+1,j,l} + 2A_{24}(k-i)(j-l+1)a_{k+1,i+1,j+1,l}\\
&+2A_{25}(k-i)(l+1)a_{k+1,i+1,j+1,l+1} + A_{33}(i-j+1)(i-j+2)a_{k+1,i+2,j,l}\\
&+2A_{34}(i-j+1)(j-l+1)a_{k+1,i+2,j+1,l} + 2A_{35}(i-j)(l+1)a_{k+1,i+2,j+1,l+1}\\
&+A_{44}(j-l+2)(j-l+1)a_{k+1,i+2,j+2,l} + 2A_{45}(j-l+1)(l+1)a_{k+1,i+2,j+2,l+1}\\
&+A_{55}(l+2)(l+1)a_{k+1,i+2,j+2,l+2} = 0
\end{aligned} \tag{2}$$

for $0 \le l \le j \le i < k \le \nu - 1$.

Let

$$L = [a_{0000}, a_{1000}, a_{1100}, a_{1110}, a_{1111}, a_{2000}, \dots, a_{\nu\nu\nu\nu}]^T$$

be the column vector, where a_{kijl} $(0 \le l \le j \le i \le k \le \nu)$ are the coefficients of polynomial $P(X)$.

Conditions (2) in matrix notation have the following form $S \cdot L = 0$, where the matrix S (the elements of this matrix are defined from conditions (2)) have the form

$$\left\|\begin{matrix}
A_{11}(\nu-1)\nu & 2A_{12}(\nu-1) & 2A_{13}(\nu-1) & 2A_{14}(\nu-1) & 2A_{15}(\nu-1) & 2A_{22} & \dots & 0\\
0 & A_{11}(\nu-1)\nu & \dots & \dots & \dots & \dots & \dots & 0\\
0 & 0 & A_{11}(\nu-1)\nu & \dots & \dots & \dots & \dots & 0\\
0 & 0 & 0 & A_{11}(\nu-1)\nu & \dots & \dots & \dots & 0\\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots\\
0 & 0 & 0 & 0 & 0 & 0 & \dots & A_{55}(\nu-1)\nu
\end{matrix}\right\|.$$

The number of rows of the matrix S is equal to $\binom{\nu+2}{4}$ and the number of columns of the matrix S is equal to $\binom{\nu+4}{4}$. Hence S is $\binom{\nu+2}{4} \times \binom{\nu+4}{4}$ matrix. We devide the matrix S into two matrices S_1 and S_2. S_1 is the left square nondegenerate $\binom{\nu+2}{4} \times \binom{\nu+2}{4}$ matrix, it consists of the first $\binom{\nu+2}{4}$ columns of the matrix S; S_2 is the right $\binom{\nu+2}{4} \times \frac{(\nu+1)(\nu+2)(2\nu+3)}{6}$ matrix, it consists of the last $\binom{\nu+4}{4} - \binom{\nu+2}{4} = \frac{(\nu+1)(\nu+2)(2\nu+3)}{6}$ columns of the matrix S.

Similarly, we devide the matrix L into two matrices L_1 and L_2. L_1 is the $\binom{\nu+2}{4} \times 1$ matrix, it consists of the upper $\binom{\nu+2}{4}$ elements of the matrix L; L_2 is the $\frac{(\nu+1)(\nu+2)(2\nu+3)}{6} \times 1$ matrix, it consists of the lower $\binom{\nu+4}{4} - \binom{\nu+2}{4} = \frac{(\nu+1)(\nu+2)(2\nu+3)}{6}$ elements of the matrix L.

According to the new notation, the matrix equality has the form $S_1L_1 + S_2L_2 = 0$, i.e. $L_1 = -S_1^{-1}S_2L_2$. It follows from this equality that the matrix L_1 is expressed through the matrix L_2, i.e., the first $\binom{\nu+2}{4}$ elements of the matrix L are expressed through its other $\frac{(\nu+1)(\nu+2)(2\nu+3)}{6} = t$ elements. Since the matrix L consists of the coefficients of the spherical polynomial $P(X)$, its first $\binom{\nu+2}{4}$ coefficients can be expressed through the last $\frac{(\nu+1)(\nu+2)(2\nu+3)}{6} = t$ coefficients.

Let $Q(X) = Q(x_1, x_2, x_3, x_4, x_5)$ be a quadratic form of five variables. We have, $\dim P(\nu, Q) = \binom{\nu+4}{4} - \binom{\nu+2}{4}$.

We have thereby proved the following theorem.

Theorem 1. *The polynomials (the coefficients of polynomial P_i are given in the brackets)*

$$P_1(a^{(1)}_{0000}, a^{(1)}_{1000}, \ldots, a^{(1)}_{\nu-2,\nu-2,\nu-2,\nu-2}, 1, 0, 0, \ldots, 0),$$

$$P_2(a^{(2)}_{0000}, a^{(2)}_{1000}, \ldots, a^{(2)}_{\nu-2,\nu-2,\nu-2,\nu-2}, 0, 1, 0, \ldots, 0),$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$P_t(a^{(t)}_{0000}, a^{(t)}_{1000}, \ldots, a^{(t)}_{\nu-2,\nu-2,\nu-2,\nu-2}, 0, 0, 0, \ldots, 1),$$

where the first $\binom{\nu+2}{4}$ coefficients from a_{0000} to $a_{\nu-2,\nu-2,\nu-2,\nu-2}$ are calculated through other $\frac{(\nu+1)(\nu+2)(2\nu+3)}{6} = t$ coefficients, form the basis of the space $P(\nu, Q)$.

Consider the generalized r-fold theta-series $\vartheta(\tau, P, Q) = \sum_{n\in\mathbb{Z}^r} P(n)z^{Q(n)},\ z = e^{2\pi i\tau}$.

We have showed [4 - 6] that, the maximal number of linearly independent theta-series for diagonal ternary quadratic forms with spherical polynomials of order ν is $\frac{\nu}{2} + 1$ and for diagonal quaternary quadratic forms is $\binom{\frac{\nu}{2}+2}{2}$. Our goal is to construct a basis of the space of generalized theta-series with spherical polynomial P of order ν and diagonal quadratic form Q of five variables.

Construct the integral automorphisms U of the diagonal quadratic form

$$Q(X) = b_{11}x_1^2 + b_{22}x_2^2 + b_{33}x_3^2 + b_{44}x_4^2 + b_{55}x_5^2.$$

An integral $r \times r$ matrix U is called an integral automorphism of the quadratic form $Q(X)$ in r variables if the condition $U^TAU = A$ is satisfied.

The integral automorphisms of the quadratic form $Q(X)$ are

$$U = \begin{Vmatrix} \pm1 & 0 & 0 & 0 & 0 \\ 0 & \pm1 & 0 & 0 & 0 \\ 0 & 0 & \pm1 & 0 & 0 \\ 0 & 0 & 0 & \pm1 & 0 \\ 0 & 0 & 0 & 0 & \pm1 \end{Vmatrix}.$$

It is known ([1], p. 37) that, if G is the set of all integral automorphisms of Q and $\sum_{i=1}^{t} P(U_iX) = 0$ for some $U_1, \ldots, U_t \in G$, then $\vartheta(\tau, P, Q) = 0$.

Consider all possible sums $\sum_{i=1}^{t} P(U_iX)=0$. For such polynomials $\vartheta(\tau,P,Q)=0$. If among the last $\frac{(\nu+1)(\nu+2)(2\nu+3)}{6}=t$ coefficients of P, at least one of four indices k,i,j,l of the coefficient, equal to one, is odd, then spherical polynomials P satisfies the equality $\vartheta(\tau,P,Q)=0$. Hence, the maximal number of linearly independent theta-series (when the indices k,i,j,l of the coefficient equal to one is even, of the corresponding spherical polynomial P) is

$$\sum_{\substack{i=0\\2|i}}^{\nu}\ \sum_{\substack{j=0\\2|j}}^{i}\ \sum_{\substack{l=0\\2|l}}^{j} 1=\sum_{\substack{i=0\\2|i}}^{\nu}\ \sum_{\substack{j=0\\2|j}}^{i}\left(\frac{j}{2}+1\right)=\sum_{\substack{i=0\\2|i}}^{\nu}\frac{(\frac{i}{2}+2)(\frac{i}{2}+1)}{2}=\binom{\frac{\nu}{2}+3}{3},$$

here $k=\nu$ is even. Thus, we have proved the following

Theorem 2. *The maximal number of linearly independent theta-series with spherical polynomial P of order ν and diagonal quadratic form Q of five variables is $\binom{\frac{\nu}{2}+3}{3}$.*

REFERENCES

1. Gooding F. Modular forms arising from spherical polynomials and positive definite quadratic forms, *J. Number Theory*, **9** (1977), 36-47.
2. Hecke E. Mathematische Werke. *Zweite Auflage, Vandenhoeck und Ruprecht, Göttingen*, 1970.
3. Lomadze G. On the basis of the space of fourth order spherical functions with respect to a positive quadratic form (Russian), *Bull. Acad. Sci. Georgian SSR*, **69** (1973), 533-536.
4. Shavgulidze K. On the dimension of some spaces of generalized ternary theta-series. *Georgian Math. J.*, **9** (2002), 167-178.
5. Shavgulidze K. On the dimension of spaces of generalized quaternary theta-series. *Proc. I. Vekua Inst. Appl. Math.*, **59-60** (2009-2010), 60-75.
6. Shavgulidze K. On the dimensions of some spaces of generalized theta-series. *Lithuanian Math. J.*, **53** (2013), 235-240.

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