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# ON NUMERICAL SOLUTION OF ANTIPLANE PROBLEMS OF ELASTICITY THEORY FOR COMPOSITE BODIES WEAKENED BY CRACKS

Papukashvili A.

**Abstract**. Antiplane problems of the elasticity theory for composity (piece-wise homogeneous) orthotropic plane is reduced to the system (pair) of singular integral equations containing an immovable singularity with respect to the tangent stress jumps (the characteristic function of the cracks expansion). In the present article the system of singular integral equations is solved by a collocation method, in particular, by discrete singular method in cases of uniform located knots. The corresponding algorithms are composed and realized. The results of theoretical and numerical investigations are presented.

**Keywords and phrases**: Singular integral equations, integral equations method, antiplane problems, cracks, collocation method.

**AMS subject classification**: 65R20, 45E05, 45F15, 65M70.

**1. Introduction.** The study of boundary value problems for the composite (piece-wise homogeneous) bodies weakened by cracks has a great practical significance. In the article [1] the antiplane problems of elasticity theory for composite (piece-wise homogeneous) orthotropic bodies weakened by cracks when cracks intersect an interface or penetrate it at rectangular angle is studied by method of integral equations. The behavior of solutions in the neighborhood of the crack end points is studied. The singular integral equations is solved by a collocation method, in particular, by discrete singular method [2]. For solution of the above mentioned problem in both cases when on the crack knots are located in the uniform as well non-uniform manner a discrete singular method is used (see [3]). In the present article the question of the approximate solution of one system (pair) of the singular integral equations is investigated. A general scheme of approximate solutions by the collocation method is composed and realized.

**2. Statement of the problem.** Let the elastic $\Omega$ body occupy the complex variable plane $z = x + iy$ which is cut on the line $L = [-1; +1]$. The plane consists of two orthotropic homogeneous semiplanes

$$\Omega_1 = \{z : Rez \geq 0,\, x \notin L_1 = [0, 1]\}$$

and

$$\Omega_2 = \{z : Rez \leq 0,\, x \notin L_2 = [-1, 0]\},$$

which are welded on the axis $y$. Define by index $k, k = 1, 2$ values and functions connected with $\Omega_k$.

The problem is to find the function $w_k(x, y)$, which satisfies the differential equation:

$$\frac{\partial^2 w_k(x,y)}{\partial x^2} + \lambda_k^2 \frac{\partial^2 w_k(x,y)}{\partial y^2} = 0, \quad (x,y) \in \Omega_k, \tag{1}$$

and boundary conditions:

a) on the boundary of the crack tangent stresses are given:

$$b_{44}^{(k)}\frac{\partial w_k(x,\pm 0)}{\partial y}=q_k^{(\pm)}(x),\quad x\in L_k, \tag{2}$$

b) on the axis $y$ the condition of continuity is fulfilled:

$$w_1(0;y)=w_2(0;y),\ y\in(-\infty;\infty),\ y\neq 0, \tag{3}$$

$$b_{55}^{(1)}\frac{\partial w_1(0;y)}{\partial x}=b_{55}^{(2)}\frac{\partial w_2(0;y)}{\partial x}, \tag{4}$$

where $\lambda_k^2=\dfrac{b_{44}^{(k)}}{b_{55}^{(k)}}$, $b_{44}^{(k)}$, $b_{55}^{(k)}$ are elastic constants, $q_k(x)$ is a function of Hölder's class, $k=1,2$; in particular, if we have an isotropic case $b_{44}^{(k)}=b_{55}^{(k)}=\mu_k,\ \lambda_k=1$, where $\mu_k$ is the module of displacement, $k=1,2$.

Using the theory of analytical functions problem (1)-(4) is reduced to the system (pair) of singular integral equations containing an immovable singularity with respect to the tangent stress jumps (the characteristic function of the cracks expansion) $\rho_k(x)$ (see[1]):

$$\begin{gathered}\int_0^1\left[\frac{1}{t-x}-\frac{a_1}{t+x}\right]\rho_1(t)dt+b_1\int_{-1}^0\frac{\rho_2(t)dt}{t-x}=2\pi f_1(x),\ \ x\in(0;1),\\ b_2\int_0^1\frac{\rho_1(t)dt}{t-x}+\int_{-1}^0\left[\frac{1}{t-x}-\frac{a_2}{t+x}\right]\rho_2(t)dt=2\pi f_2(x),\ \ x\in(-1;0),\end{gathered}\tag{5}$$

where $\rho_k(x)$, $f_k(x)$ are unknown and given real functions, respectively, $a_k,b_k$ are constants

$$a_k=\frac{1-\gamma_k}{1+\gamma_k},\quad b_k=\frac{2}{1+\gamma_k},\quad \gamma_1=1/\gamma_2,\quad \gamma_2=\frac{b_{55}^{(2)}}{b_{55}^{(1)}},$$

$$f_k(x)=\frac{\lambda_k}{b_{44}^{(k)}}q_k(x),\quad f_k(x)\in H,\quad \rho_k(x)\in H^*,\ k=1,2.$$

**3. The algorithm.** The system of singular integral equations (5) is solved by a collocation method, in particular, by discrete singular method (see[3]) in cases of both uniform, and non-uniformly located knots. In particular, in this article algorithm of uniform splitting of an interval is used.

Decisions of the system of equations (5) of this form $\rho_1(t)=\dfrac{\rho_1^*(t)}{\sqrt{1-t}}$, $\rho_2(t)=\dfrac{\rho_2^*(t)}{\sqrt{1+t}}$, (see [1], [3]).

$$\rho_1(t),\,\rho_2(t)\in H^*,\quad \rho_1^*(t),\,\rho_2^*(t)\in H.$$

Let's enter such distribution of knots for variables of integration and account points accordingly

$$t1_i=0+ih,\quad t2_i=-1+ih,\quad i=1,2,\cdots,n;$$

$$x1_j=t1_j-h/2,\quad x2_j=t2_j+h/2,\quad j=1,2,\cdots,n;$$

$$h=\frac{1}{n+1}.$$

The pair of equations (5) is probably presented as follows with the help of quadrature formulas

$$\sum_{i=1}^{n}\left(\frac{h}{t1_i - x1_j} - \frac{a_1 h}{t1_i + x1_j}\right)\rho_1(t1_i)$$

$$+b_1\sum_{i=1}^{n}\left(\frac{h}{t2_i - x1_j}\right)\rho_2(t2_i) = 2\pi f_1(x1_j), \quad j = 1, 2, \cdots, n;$$

$$b_2\sum_{i=1}^{n}\left(\frac{h}{t1_i - x2_j}\right)\rho_1(t1_i) \tag{6}$$

$$+\sum_{i=1}^{n}\left(\frac{h}{t2_i - x2_j} - \frac{a_2 h}{t2_i + x2_j}\right)\rho_2(t2_i) = 2\pi f_2(x2_j), \ \ j = 1, 2, \cdots, n.$$

Thus, we have $2n$ equations with $2n$ unknowns. It is possible to solve the obtained system of linear equations with the help of one of the direct methods for example, by Gauss method.

**4. Numerical experiments and discussion of results.** As it has been mentioned, from the system of algebraic equations (6) numerical values of unknown characteristic function of disclosing of cracks in $2n$ central points are defined. The base system of the integral equation (5) is solved by a discrete singular method, when $a_1 = a_2 = 0$, $b_1 = b_2 = 0$,

$f1(x) = 1, \quad f2(x) = 1$ (case I);
$f1(x) = 1, \quad f2(x) = 2$ (case II);
$f1(x) = 2, \quad f2(x) = 1$ (case III).

In the above - stated problems a main objective was to study behavior of solution (characteristic function of expansion of a crack), calculation of coefficients of intensity of stress ($cis_1$,$cis_2$) on the crack ends, research of behavior of distribution of cracks (the forecast of distribution of a crack).

In the above-mentioned research tasks a main objective is to study behavior of solution in the vicinity at the ends of cracks and finding the coefficients of intensity of stress

$$cis_1 = \lim_{x\to -1}\sqrt{1+x}\rho_2(x), \quad cis_2 = \lim_{x\to +1}\sqrt{1-x}\rho_1(x)$$

on the ends of cracks.

For this purpose the values of the coefficients of intensity of stress (stress intensity factor)

$$cis_1 \approx \sqrt{1+x2_1}\rho_2(x2_1), \quad cis_2 \approx \sqrt{1-x1_n}\rho_1(x1_n)$$

were calculated by algorithm of uniform splitting of the interval $[-1, 1]$ accompanied by the increment of the number of splitting two times at each step of the calculations.

The approached values of coefficients of intensity of stresses in vicinity of the ends of a crack (at constant and uniform loadings)

$f1(x) = 1, \quad f2(x) = 1$ variant 1,
$f1(x) = 1, \quad f2(x) = 2$ variant 2,

$f1(x) = 2, \quad f2(x) = 1$ variant 3 are given by the Table 1.

| v | n | 16 | 32 | 64 | 128 | 256 | 512 | 1024 | 2048 | 4096 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | $cis_1$ | 2.422 | 2.464 | 2.485 | 2.496 | 2.501 | 2.504 | 2.505 | 2.506 | 2.506 |
| | $cis_2$ | -2.422 | -2.464 | -2.485 | -2.496 | -2.501 | -2.504 | -2.505 | -2.506 | -2.506 |
| 2 | $cis_1$ | -0.607 | -1.137 | -1.681 | -2.231 | -2.784 | -3.337 | -3.890 | -4.443 | -4.996 |
| | $cis_2$ | -7.873 | -8.528 | -9.136 | -9.716 | -10.287 | -10.848 | -11.406 | -11.961 | -12.515 |
| 3 | $cis_1$ | 7.873 | 8.528 | 9.136 | 9.716 | 10.287 | 10.848 | 11.406 | 11.961 | 12.515 |
| | $cis_2$ | 0.607 | 1.137 | 1.681 | 2.231 | 2.784 | 3.337 | 3.890 | 4.443 | 4.996 |

Table 1.

Performed calculations show that if the absolute values of the coefficients of intensity of tension are less than 1 (very close to critical limit of distribution of a crack) but close to unit then cracks are developed very slowly, i.e. almost are not developed (see [4]). Numerical experiments also show that increment of loading at the ends of the crack causes increment of values of the coefficients of intensity of tension. As we consider linear tasks of the elasticity theory the increment or diminution loading will lead to proportional increment or diminution of values of relevant solutions. The last fact gives possibility to make the hypothetical forecasts about developments of a crack.

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Author's address:

A. Papukashvili
Iv. Javakhishvili Tbilisi State University
I. Vekua Institute of Applied Mathematics
& Faculty of Exact and Natural Sciences
2, University St., Tbilisi 0186
Georgia
E-mail: apapukashvili@rambler.ru,
archil.papukashvili@tsu.ge