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SOLOVAY MODEL AND DUALITY PRINCIPLE BETWEEN THE MEASURE AND THE BAIRE CATEGORY IN A POLISH TOPOLOGICAL VECTOR SPACE $H(X, S, \mu)$

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Abstract. In Solovay model it is shown that the duality principle between the measure and the Baire category holds true with respect to the sentence -"The domain of an arbitrary generalized integral for a vector-function is the set of first category".

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1. Introduction. In [1] it has been shown that the existence of a non-Lebesgue measurable set cannot be proved in Zermelo-Frankel set theory ZF if use of the axiom of choice is not allowed. In fact, even adjoining an axiom DC to ZF, which allows countably many consecutive choices, does not create a theory strong enough to construct a non-measurable set. S. Shelah [2] proved that an inaccessible cardinal is necessary to build a model of set theory in which every set of reals is Lebesgue measurable. A simpler and metamathematically free proof of Shelah's result was given in [3]. As a corollary, it was proved that the existence of an uncountable well ordered set of reals implies the existence of a non-measurable set of reals.

Here are several thousand papers devoted to investigations in Solovay model. For example, in [4] has been proved that in such a model each linear operator on a Hilbert space is a bounded linear operator. It was shown in [5] that the domain of an arbitrary generalized integral in the same model is the set of the first category. In [6], an example of a non-zero non-atomic translation-invariant Borel measure ν_p on the Banach space $l_p(1 \leq p < \infty)$ is constructed in Solovay's model such that the condition "ν_p-almost every element of l_p has a property P" implies that "*almost every element of* l_p (in the sense of [7]) has the property P".

In order to formulate the main goal of the present paper, we introduce a new approach which can be considered as a certain modification of the *duality principle between the measure and the Baire category* introduced in [8]: Let (E, τ) be a topological vector space. Denote by $B(E)$ the Borel σ-algebra of subsets of the space E. A universally measurable subset of A is called small in the sense of measure if A is shy set. Analogously, a subset $B \subseteq E$ is called small in the sense of category if it is of first category in E. Further, let P be a sentence formulated only by using the notions of measure zero, of the first category and of purely set-theoretical notions. We say that the duality principle between the measure and the Baire category holds true with respect to the sentence P if the sentence P is equivalent to the sentence P^* obtained from the sentence P by interchanging in it the notions of the above small sets.

The present paper is devoted to study of the question asking whether the duality principle holds true between the measure and the Baire category with respect to the sentence - "*The domain of an arbitrary generalized integral for a vector-function is the set of the first category.*"

The paper is organized as follows.

In Section 2 we consider some auxiliary notions and facts from the measure theory and topology. In Section 3 we give the proof of the main result.

2. Some auxiliary notions and facts from measure theory and topology

Definition 2.1. Let (E,τ) be a topological space. We say that a subset $Y \subseteq E$ has Baire property if Y admits the following representation

$$Y = (Z \cup Z^{'}) \setminus Z^{''},$$

where Z is an open set in E, $Z^{'}$ and $Z^{''}$ are sets of the first category.

Definition 2.2. We say that a topological space (E,τ) is Baire space if an arbitrary non-empty open subset of E is the set of the second category.

Remark 2.1. Following Solovay [1], let 'ZF' denote the axiomatic set theory of Zermelo-Fraenkel and let 'ZF + DC' denote the system obtained by adjoining a weakened form of the axiom of choice, DC, (see p. 52 of [1] for a formal statement of DC). The system ZF + DC is important because all the positive results of elementary measure theory and most of the basic results of elementary functional analysis, except for the Hahn-Banach theorem and other such consequences of the axiom of choice, are provable in ZF + DC. In particular, the Baire category theorem for complete metric spaces and the closed graph theorem for operators between Fréchet spaces are provable in ZF + DC.

Let ZFC be Zermelo-Frankel set theory together with the axiom of choice. In the presence of the axiom of choice, we identify cardinals with initial ordinals. A cardinal $\aleph$ is regular, if each order unbounded subset of $\aleph$ has power $\aleph$. A cardinal $\aleph$ is inaccessible, if it is regular, uncountable, and for $\aleph^{'} < \aleph$, $2^{\aleph^{'}} < \aleph$.

Let I be the statement: "*There is an inaccessible cardinal*".

Lemma 2.1. ([1], Theorem 1, p.1) *Suppose that there is a transitive ϵ-model of ZFC + I. Then there is a transitive ϵ-model of ZF in which the following pro positions are valid.*

(1) The principle of dependent choice (DC).

(2) Every set of reals is Lebesgue measurable.

(3) Every set of reals has the property of Baire.

(4) Every uncountable set of reals contains a perfect subset.

(5) Let $\{A_x : x \in R\}$ be an indexed family of non-empty set of reals with index set of reals. Then there are Borel functions h_1, h_2 mapping R into R such that

(a) $\{x : h_1(x) \notin A_x\}$ has Lebesgue measure zero.

(b) $\{x : h_2(x) \notin A_x\}$ is of first category.

Let (X, S, μ) be a probability space and let H be a separable Banach space over the vector field R. Let us denote by $H(X, S, \mu)$ a class of all μ-measurable mappings of the X into H. As usual, we will identify equivalent mappings, i.e., we will consider the factor-space with respect to an equivalent relation on $H(X, S, \mu)$. If μ is separable and the metric ρ in $H(X, S, \mu)$ is defined by

$$\rho(f, g) = \int_X ||f - g||/(1 + ||f - g||)d\mu,$$

then $H(X, S, \mu)$ stands for a Polish topological vector space without isolated points. Suppose that μ is separable and diffused. Let M be a vector subspace of $H(X, S, \mu)$ which contains the class of all measurable step functions. A linear operator $I : M \to H$ is called a generalized integral for a vector-function if for each step function $g : X \to H$ we have

$$I(g) = \sum_{i=1}^{n} a_i \mu(X_i),$$

where $(X_i)_{1\leq i\leq n}$ is a measurable partition of X and g takes the value a_i at points of X_i for each $i(1 \leq i \leq n)$. It is obvious that Reiman, Lebesgue and Denjoy integrals are partial cases of the generalized integral for a vector-function and they are realized when under H is taken the real axis R considered as a Banach space over vector field R.

Lemma 2.2. ([5], Theorem, p. 34) *Let as denote by (SM) the transitive ϵ-model of ZF which comes from Lemma 2.1. Then in (SM) the domain of an arbitrary generalized integral for a vector-function is of first category in $H(X,S,\mu)$.*

Definition 2.3 Let $\mathbb{V}$ be a Polish topological vector space. Let K be the class of all probability diffused Borel measures defined on the Borel σ-algebra $\mathcal{B}(\mathbb{V})$. We denote by $\mathcal{B}(\mathbb{V})^{\mu}$ the completion of $\mathcal{B}(\mathbb{V})$ with respect to the measure μ for $\mu \in K$. A set $E \subset \mathbb{V}$ is called universally measurable if $E \in \cap_{\mu\in K}\mathcal{B}(\mathbb{V})^{\mu}$.

Lemma 2.3. ([9], Remark 3.4., p. 40) *(SM) Let $\mathbb{V}$ be a complete separable metric space. Then every subset of $\mathbb{V}$ is universally measurable.*

Definition 2.3. ([7], Definition 1, p. 221) A measure μ is said to be transverse to a universally measurable set $S \subset \mathbb{V}$ if the following two conditions hold:

(i) There exists a compact set $U \subset \mathbb{V}$ for which $0 < \mu(U) < 1$.

(ii) $\overline{\mu}(S+v) = 0$ for every $v \in \mathbb{V}$, where $\overline{\mu}$ denotes a usual completion of the measure μ.

Definition 2.3. ([7], Definition 2, p. 222) A universally measurable set $S \subset \mathbb{V}$ is called shy if there exists a measure transverse to S. More generally, a subset of $\mathbb{V}$ is called shy if it is contained in a universally measurable shy set. The complement of a shy set is called a prevalent set.

Remark 2.1. Note that the class of all shy sets in a Polish topological vector space $\mathbb{V}$ is a σ-ideal (see, [7], Fact $3''$, p. 223).

3. Main results.

Theorem 3.1. *(SM) The domain of an arbitrary generalized integral for a vector-function in $H(X,S,\mu)$ is shy.*

Proof. Let I be a generalized integral for a vector-function in $H(X,S,\mu)$ and let M be its domain. Since $H(X,S,\mu)$ is a Polish topological vector space, by Lemma 2.3 we know that M is universally measurable. By Lemma 2.2 we know that M is the set of the first category in $H(X,S,\mu)$. Following Remark 2.1, $H(X,S,\mu)$ is the Baire space which implies that $H(X,S,\mu) \setminus M \neq \emptyset$. Let $v \in H(X,S,\mu) \setminus M$. Let us show that v spans a line L such that every translate of L meets M in at most one point. Indeed, assume the contrary. Then there will be an element $z \in H(X,S,\mu)$ and two different parameters $t_1, t_2 \in \mathbb{R}$ such that $z + t_1v \in M$ and $z + t_2v \in M$. Since M is a vector subspace of $H(X,S,\mu)$ we deduce that $(t_2 - t_1)v \in M$. Using the same argument we claim that $v \in M$ because $t_2 - t_1 \neq 0$, but the latter relation is the contradiction. Hence, the Lebesgue measure supported on L is transverse to a universally measurable set M which means that M is shy.

By using Lemma 2.2 and Theorem 3.1 we get the validity of the following assertion.

Theorem 3.2 *(SM) The duality principle between the measure and the Baire category is valid with respect to the sentence P defined by*

"The domain of an arbitrary generalized integral for a vector-function is the set of the first category in $H(X,S,\mu)$."

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