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NEUTRAL SURFACES OF A NON-SHALLOW SHELLS

Meunargia T.

Abstract. In the paper the problem of existence of neutral surfaces for non-shallow shell is considered. By neutral surface is meant a surface which is not subject to tensions and compression under the deformation on the shell. It means that the neutral surfaces may be subject only to bendings or, in particular, may remain rigid. I. Vekua obtained the conditions for the existence of neutral surface of a shell, when the neutral surface is the middle surface. In this paper the neutral surface is considered as any equidistant surfaces.

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1. Complete system of equilibrium equations and the stress-strain relations of the 3-D non-shallow shells can be written as

$$\frac{1}{\sqrt{g}}\partial_i(\sqrt{g}\boldsymbol{\sigma}^i)+\boldsymbol{\Phi}=0,\quad \boldsymbol{\sigma}^i=\sigma^i_j\boldsymbol{R}^j=(\lambda\theta g^i_j+2\mu e^i_j)\boldsymbol{R}^j,\quad (i,j=1,2,3) \tag{1}$$

where g is the discriminant of the metric quadratic form of the 3-D domain, σ^i_j and e^i_j are the mixed components of stress and strain tensors, $\boldsymbol{\Phi}$ is an external force, λ and μ are Lame's constants, $g^i_j=\boldsymbol{R}_j\boldsymbol{R}^i$, $\boldsymbol{R}_j$ and $\boldsymbol{R}^i$ are covariant and contravariant of bases vectors of the surface $s(x^3=\text{const})$, θ is the cubical dilatation which will be written as

$$\theta=\theta'+e^3_3,\quad \theta'=e^\alpha_\alpha\ \ (\alpha=1,2). \tag{2}$$

Further

$$\boldsymbol{R}_\alpha=(a^\beta_\alpha-x^3b^\beta_\alpha)\boldsymbol{r}_\beta,\quad \boldsymbol{R}^\alpha=\vartheta^{-1}[a^\alpha_\beta+x^3(b^\alpha_\beta-2Ha^\alpha_\beta)]\boldsymbol{r}^\beta,$$
$$\boldsymbol{R}_3=\boldsymbol{R}^3=\boldsymbol{n},\vartheta=1-2Hx_3+Kx_3^2,\quad (\beta=1,2),$$

where a^β_α and b^β_α are the mixed components of the metric and curvatures tensors of the middle surface $S:(x^3=0)$, H and K are, respectively, middle and Gaussian curvatures of the surface S.

When $j=3$ from (1) we have

$$\begin{aligned}&\sigma^\alpha_3=2\mu e^\alpha_3,\quad \sigma^3_3=\lambda\theta+2\mu e^3_3=\lambda\theta'+(\lambda+2\mu)e^3_3\\ &\Rightarrow e^\alpha_3=\frac{1}{2\mu}\sigma^\alpha_3,\quad e^3_3=-\frac{\lambda}{\lambda+2\mu}\theta'+\frac{1}{\lambda+2\mu}\sigma^3_3.\end{aligned} \tag{3}$$

By inserting expression (2) into (3) we obtain

$$\theta=\frac{\lambda'}{\lambda}\theta'+\frac{1}{\lambda+2\mu}\sigma^3_3,\quad \Big(\lambda'=\frac{2\lambda\mu}{\lambda+2\mu}\Big).$$

Substituting expression (3) into (1) we get

$$\sigma^i_j = T^i_j + Q^i_j,\ T^\alpha_\beta = \lambda'\theta g^\alpha_\beta + 2\mu e^\alpha_\beta,\ Q^\alpha_\beta = \sigma'\sigma^3_3 g^\alpha_\beta,\ T^i_3 = 0,\ Q^i_3 = \sigma^i_3\ \left(\sigma' = \frac{\lambda}{\lambda + 2\mu}\right) \quad (4)$$

$$\Rightarrow e^\alpha_\beta = \frac{1}{2\mu}T^\alpha_\beta - \frac{\lambda'}{u_\mu(\lambda' + \mu)}T^\gamma_\gamma g^\alpha_\beta,\ \ \theta' = \frac{1}{2(\lambda' + \mu)}T^\gamma_\gamma.$$

If we now insert (1) into the formula $\boldsymbol{\sigma}^i = \sigma^i_j \boldsymbol{R}^j$ expressing the contravariant components of the stress field, we obtain

$$\boldsymbol{\sigma}^\alpha = \boldsymbol{T}^\alpha + \boldsymbol{Q}^\alpha,\ \ \boldsymbol{\sigma}^3 = \boldsymbol{Q}^3 \Rightarrow \boldsymbol{T}^\alpha = T^\alpha_\beta \boldsymbol{R}^\beta,\ \ \boldsymbol{T}^3 = 0,\ \ \boldsymbol{Q}^i = Q^i_j \boldsymbol{R}^j. \quad (5)$$

Formula (4) implies that the vector $\boldsymbol{T}^\alpha$ satisfies the condition

$$\boldsymbol{n}\boldsymbol{T}^\alpha = 0.$$

The stress field expressed by the vector $\boldsymbol{T}^\alpha$, is therefore called the tangential stress field and vector $\boldsymbol{Q}^i$ be called the transverse field.

2. In view of (5) the vectorial equation of equilibrium

$$\frac{1}{\sqrt{g}}\partial_i(\sqrt{g}\boldsymbol{\sigma}^i) + \boldsymbol{\Phi} = 0, \quad (\sqrt{g} = \sqrt{a}\vartheta),$$

may be written as

$$\frac{1}{\sqrt{g}}[\partial_\alpha(\sqrt{g}\boldsymbol{T}^\alpha) + \partial_i(\sqrt{g}\boldsymbol{Q}^i)] + \boldsymbol{\Phi} = 0. \quad (6)$$

Let the surface $\hat{S} : x^3 = const$ be the neutral surface of a non-shallow shell. Then

$$\boldsymbol{T}^\alpha = 0,\ \text{ i.e. } T^{\alpha\beta} = 0\ \ (\text{on } \hat{S}),$$

and equation (6) becomes

$$\left[\frac{1}{\sqrt{g}}\partial_\alpha(\sqrt{g}\boldsymbol{Q}^\alpha) + \frac{1}{\sqrt{g}}\partial_3(\sqrt{g}\boldsymbol{\sigma}^3) + \boldsymbol{\Phi}\right]_{x^3=\text{const}} = 0, \quad (-h \leq x^3 = x_3 \leq h), \quad (7)$$

where $2h$ is the thickness of shell and

$$\boldsymbol{Q}^\alpha = Q^\alpha_\beta \boldsymbol{R}^\beta + Q^\alpha_3 \boldsymbol{n} = \sigma'\sigma^3_3 \boldsymbol{R}^\alpha + \sigma^\alpha_3 \boldsymbol{n}.$$

Denote the stress forces acting on the face surfaces S^+ and S^- by $\overset{(+)}{\boldsymbol{P}}$ and $\overset{(-)}{\boldsymbol{P}}$. We have

$$\overset{(+)}{\boldsymbol{P}} = -(\boldsymbol{\sigma}^3)_{x^3=h},\ \ \overset{(-)}{\boldsymbol{P}} = (\boldsymbol{\sigma}^3)_{x^3=-h}. \quad (8)$$

If we approximately represent $\boldsymbol{\sigma}^3$ by the formula

$$\boldsymbol{\sigma}^3(x^1, x^2, x^3) \cong \overset{0}{\boldsymbol{\sigma}}(x^1, x^2) + x^3\,\overset{1}{\boldsymbol{\sigma}}(x^1, x^2),$$

from (8) is obtained

$$\boldsymbol{\sigma}^3(x^1,x^2,x^3) \cong \frac{1}{2}\Big[\overset{(+)}{\boldsymbol{P}} - \overset{(-)}{\boldsymbol{P}} + \frac{x^3}{h}\Big(\overset{(+)}{\boldsymbol{P}} + \overset{(-)}{\boldsymbol{P}}\Big)\Big]. \tag{9}$$

In view of equalities (4) and (9) we may write equation (7) as

$$\left\{\frac{1}{\sqrt{a}}\partial_\alpha\Big[\sqrt{a}\Big(\sigma' A^\alpha_\beta\Big(\overset{(+)}{P^3} - \overset{(-)}{P^3}\Big)\boldsymbol{r}^\beta + A\Big(\overset{(+)}{P^\alpha} - \overset{(-)}{P^\alpha}\Big)\boldsymbol{n}\Big)\Big] + B\Big(\overset{(+)}{\boldsymbol{P}} - \overset{(-)}{\boldsymbol{P}}\Big) - \tilde{\boldsymbol{\Phi}}\right\}_{x^3=c} = 0, \tag{10}$$

where

$$\begin{aligned}
&A^\alpha_\beta = \Big(1+\frac{c}{h}\Big)[a^\alpha_\beta + c(b^\alpha_\beta - 2Ha^\alpha_\beta)], A = \Big(1+\frac{c}{h}\Big)(1-2Hc+kc^2),\\
&B = \frac{1}{h}[1-2Hh+2(kh-2H)c+3kc^2],\\
&\tilde{\boldsymbol{\Phi}} = \frac{1}{h}[2(1-2Hc)+kc^2]\overset{(-)}{\boldsymbol{P}} - 2(\vartheta\boldsymbol{\Phi})_{x^3=c}.
\end{aligned} \tag{11}$$

Thus, if the surface $x^3 = const$ is neutral the stress $\overset{(+)}{\boldsymbol{P}}$ and $\overset{(+)}{\boldsymbol{P}}$, applied to the face surfaces, must satisfy the vector equation (10). This means that the stresses $\overset{(+)}{\boldsymbol{P}}$ and $\overset{(-)}{\boldsymbol{P}}$ cannot be prescribed arbitrarily. It will be show that if one of these vectors is given, the other may be defined by equation (10). For example, in aircraft or submarine apparatus the force $\overset{(-)}{\boldsymbol{P}}$ acting on the inner surface S^- may be assumed to be prescribed, but the $\overset{(+)}{\boldsymbol{P}}$ acting on the external face surface S^+ is not, in general, assigned beforehand. The same situation occurs on dams. One face surface of the dam is free from stresses (or rather, is subject to atmospheric pressure), and the other is under the hydrodynamic load, a variable which is generally difficult to define exactly at any moment of time.

In future therefore only one of these forces is assumed to be prescribed, for example $\overset{(-)}{\boldsymbol{P}}$ and we try to define the second by means of equation (10).

To define the vector field $\overset{(+)}{\boldsymbol{P}}$ we have the equation

$$\left\{\frac{1}{\sqrt{a}}\partial_a[\sqrt{a}(\sigma' A^\alpha_\beta p\boldsymbol{r}^\beta + Ap^\alpha\boldsymbol{n})] + B(p\boldsymbol{n} + p^\alpha\boldsymbol{r}_\alpha) + \tilde{\boldsymbol{\Phi}}\right\}_{x^3=c} = 0, \tag{12}$$

where

$$p = \overset{(+)}{P^3} - \overset{(-)}{P^3}, \;\; p^\alpha = \overset{(+)}{P^\alpha} - \overset{(-)}{P^\alpha}.$$

Let the middle surface $S : x^3 = 0$ be the neutral surface of a shell (I. Vekua). Then (11) becomes:

$$A^\alpha_\beta = a^\alpha_\beta, \;\; A = 1, \;\; B = \frac{1}{h}(1-2hH), \;\; \tilde{\Phi} = -2\Phi(x^1,x^2,0),$$

and equation (12) takes the form

$$\frac{1}{\sqrt{a}}[\partial_\alpha \sqrt{a}(\sigma' p \boldsymbol{r}^\alpha + p^\alpha \boldsymbol{n})] + \frac{1}{h}(1 - 2hH)(p\boldsymbol{n} + p^\alpha \boldsymbol{r}_\alpha) + \tilde{\boldsymbol{\Phi}}(x^1, x^2, 0) = 0.$$

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Author's address:

T. Meunargia
I. Vekua Institute of Applied Mathematics of
Iv. Javakhishvili Tbilisi State University
2, University St., Tbilisi 0186
Georgia
E-mail: tengiz.meunargia@viam.sci.tsu.ge