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ON ONE TWO-DIMENSIONAL MODEL BASED ON MAXWELL'S SYSTEM

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Abstract. One two-dimensional nonlinear partial integro-differential equation is considered. The model is based on Maxwell's system which arises at describing penetration of a magnetic field into a substance. The initial-boundary value problem with homogeneous boundary conditions is considered. Large time behavior of solution of the initial-boundary value problem is studied. Rate of stabilization is given.

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Mathematical modeling of diffusion of a magnetic field into a substance whose electric conductivity depends on temperature is described by system of Maxwell's equations [1]. If the coefficient of thermal heat capacity and electroconductivity of the substance depend on temperature, then Maxwell's system can be reduced to the following integro-differential form [2]:

$$\frac{\partial H}{\partial t} = -rot\left[a\left(\int_0^t |rotH|^2 \, d\tau\right) rotH\right], \tag{1}$$

where $H = (H_1, H_2, H_3)$ is a vector of the magnetic field and function $a = a(S)$ is defined for $S \in [0, \infty)$.

Note that integro-differential model (1) is complex and still yields to the investigation for special cases (see, for example, [2] - [10] and references therein).

Let us consider one component and two dimensional magnetic field, i.e., assume that the vector of magnetic field has the following form $H = (0, 0, U)$ and $U = U(x, y, t)$. Then we have

$$rotH = \left(\frac{\partial U}{\partial y},\ -\frac{\partial U}{\partial x},\ 0\right)$$

and

$$rot\left(rot\left(a(S)\frac{\partial U}{\partial y},\ -a(S)\frac{\partial U}{\partial x},\ 0\right)\right) = \left(0,\ 0,\ -\frac{\partial}{\partial x}\left(a(S)\frac{\partial U}{\partial x}\right) - \frac{\partial}{\partial y}\left(a(S)\frac{\partial U}{\partial y}\right)\right),$$

where

$$S(x, y, t) = \int_0^t \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial U}{\partial y}\right)^2\right] d\tau.$$

Therefore, (1) takes the following form

$$\frac{\partial U}{\partial t}=\frac{\partial}{\partial x}\left[a\left(\int\limits_0^t\left[\left(\frac{\partial U}{\partial x}\right)^2+\left(\frac{\partial U}{\partial y}\right)^2\right]d\tau\right)\frac{\partial U}{\partial x}\right]$$
$$+\frac{\partial}{\partial y}\left[a\left(\int\limits_0^t\left[\left(\frac{\partial U}{\partial x}\right)^2+\left(\frac{\partial U}{\partial y}\right)^2\right]d\tau\right)\frac{\partial U}{\partial y}\right]. \tag{2}$$

In [9] some generalization of the system of type (1) is proposed. In particular, assuming the temperature of the considered body to be constant throughout the material, i.e., depending on time, but independent of the space coordinates, the process of penetration of the magnetic field into the material is modeled by, so-called, averaged integro-differential model, (2) type analog of which has the following form

$$\frac{\partial U}{\partial t}=a\left(S\right)\left(\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}\right), \tag{3}$$

where

$$S(t)=\int\limits_0^t\int\limits_\Omega\left[\left(\frac{\partial U}{\partial x}\right)^2+\left(\frac{\partial U}{\partial y}\right)^2\right]dxdyd\tau \tag{4}$$

and $\Omega=[0,\,1]\times[0,\,1]$.

Many scientific works are dedicated to the investigations of one-dimensional case of (1) type models (see, for example, [2] - [10] and references therein). The existence and uniqueness of the solutions of the initial-boundary value problems for the one-dimensional analog of (2) and (3), (4) type models are studied in a number of works (see, for example, [2], [3], [5], [9] and reference therein). Questions of existence and uniqueness of solutions in multi-dimensional case for (1) type models are considered in [4], [9] and for averaged (3), (4) type models in [5]. Asymptotic behavior of solutions and issues of approximate solutions are considered in many works as well (see, for example, [5] - [8] and references therein).

Our aim is to study the asymptotic behavior of solutions as $t\to\infty$ for the Dirichlet initial-boundary value problem with homogeneous boundary conditions for two-dimensional equation (3), (4). Attention will be paid to the case $a(S)=(1+S)^p$, $0<p\le 1$.

Therefore, let us consider the following initial-boundary value problem:

$$\frac{\partial U}{\partial t}=a(S)\left(\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}\right),\quad (x,y,t)\in\Omega\times(0,\,\infty), \tag{5}$$

$$U(x,y,t)=0,\quad (x,y)\in\partial\Omega,\quad t\ge 0, \tag{6}$$

$$U(x,y,0)=U_0(x,y),\quad (x,y)\in\Omega, \tag{7}$$

where

$$S(t)=\int\limits_0^t\int\limits_\Omega\left[\left(\frac{\partial U}{\partial x}\right)^2+\left(\frac{\partial U}{\partial y}\right)^2\right]dxdyd\tau, \tag{8}$$

$a(S) = (1+S)^p, 0 < p \leq 1$ and $U_0 = U_0(x,y)$ is a given function, $\partial\Omega$ is a boundary of Ω.

The following theorem takes place.

Theorem. *If $a(S) = (1+S)^p$, $0 < p \leq 1$; $U_0 \in H_0^1(0,1)$, then for the solution of problem (5)-(8) the following estimate is true*

$$\left\| \frac{\partial U}{\partial x} \right\| + \left\| \frac{\partial U}{\partial y} \right\| \leq C \exp\left(-t\right).$$

Here $H_0^1(0,1)$ is the Sobolev space, norm is an usual norm of space $L_2(\Omega)$ and C is positive constant independent of t.

Numerous numerical experiments for different initial and boundary data are fulfilled. All experiments were carried out by using software FreeFem++ [11]. In pictures (Figs. 1 and 2) below there are numerical solutions for the two-dimensional equation (5) with homogeneous Dirichlet boundary conditions. From these figures can be deduced that when time is increasing solution is vanishing, that was shown theoretically in the theorem above.

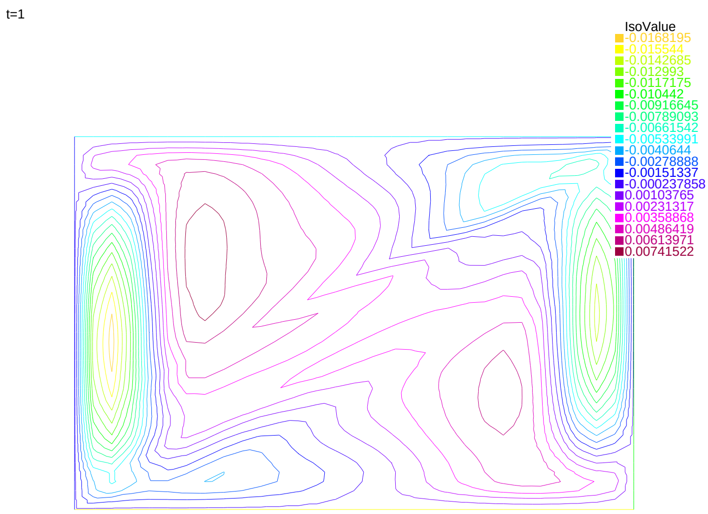

Fig.1. The numerical solution u at $t = 1$.

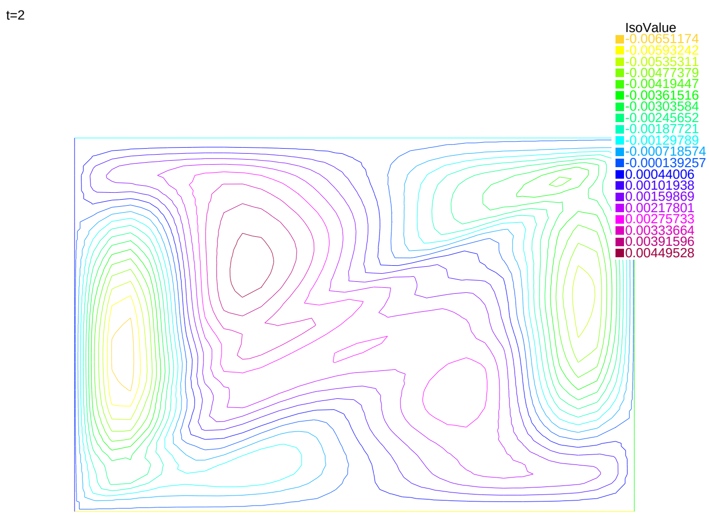

Fig.2. The numerical solution u at $t = 2$.

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