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MODIFIED KdV EQUATION FOR MAGNETIZED ROSSBY WAVES IN A ZONAL FLOW OF THE IONOSPHERIC E-LAYER

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Abstract. Nonlinear interaction of magnetized Rossby waves with sheared zonal flow in the Earth's ionospheric E-layer is investigated. It is shown that in case of weak nonlinearity 2D Charney vorticity equation can be reduced to the one-dimensional modified KdV equation.

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Introduction. Nonlinear interaction of sheared along the meridians zonal flows with the accompanied ultra-low-frequency (ULF) planetary-scale wave perturbations governs the global atmospheric circulation in the E- and F-layers of the Earth's ionosphere (see e.g. [1,2]). Essential contribution in generation of Rossby-Khantadze electromagnetic planetary vortices by shear flow in the ionospheric E-layer was recently done by Kaladze et al. [3].

In the given paper we consider the interaction of sheared along the meridians zonal flow with magnetized Rossby waves [4] in the Earth's ionospheric E-layer and show that in case of weak nonlinearity 2D Charney vorticity equation can be reduced to one-dimensional Korteweg-de Vries (KdV) equation.

Basic equation. Initial nonlinear equation describing the propagation of magnetized Rossby waves in the Earth's ionospheric E-layer is the following dimensionless modified Charney equation [4]

$$\frac{\partial \triangle\psi}{\partial t} + J(\psi, \triangle\psi) + \beta_G \frac{\partial \psi}{\partial x} = 0. \tag{1}$$

Here $\psi(x, y, t)$ is the stream function, the nonlinear term is presented by the Jacobian $J(a,b) = \dfrac{\partial a}{\partial x}\dfrac{\partial b}{\partial y} - \dfrac{\partial a}{\partial y}\dfrac{\partial b}{\partial x}$, $\triangle = \partial_x^2 + \partial_y^2$ is the two-dimensional Laplacian,and

$$\beta_G(y) = \frac{\partial}{\partial y}\left(2\Omega_{0z}(y) + \frac{en}{\rho} B_{0z}(y)\right) \tag{2}$$

is the generalized Rossby parameter. Eq. (2) describes latitudinal (over the coordinate y) variation of the Coriolis parameter $\Omega_{0z}(y)$ and the geomagnetic field $B_{0z}(y)$; n is the charged particles number density and ρ is the mass density of neutrals. As the mean zonal flow exists between southern and northern latitudinal boundaries we impose here the following rigid-well boundary conditions on the variable y

$$\psi(0) = \psi(1) = 0. \tag{3}$$

We should emphasize that in contrast to [5] we don't use the β- plane approximation here and consider the expression (2) in general without expansion in a series over the latitude coordinate y.

Derivation of the modified KdV equation. We represent the total stream function ψ as the sum of the basic zonal flow $\Psi(y)$ and the perturbed stream ψ' with a non-dimensional amplitude ϵ, i.e.

$$\psi(x, y, t) = \Psi(y) + \epsilon\psi'(x, y, t). \tag{4}$$

We assume here that the basic stream function $\Psi(y)$ has the form

$$\Psi(y) = -\int [U(y) - c_0] dy, \tag{5}$$

where c_0 is a eigenvalue of the problem set below, and $U(y)$ is the basic flow.

We will consider $\epsilon << 1$, which means weakly nonlinear problem we are engaged here. Substitution of Eqs. (4) and (5) into Eq. (1) gives

$$\frac{\partial \triangle\psi}{\partial t} + (U - c_0)\frac{\partial \triangle\psi}{\partial x} + p(y)\frac{\partial\psi}{\partial x} + \epsilon J(\psi, \triangle\psi) = 0. \tag{6}$$

Here the apostrophe of the disturbance stream is dropped, and the following function was introduced

$$p(y) = \beta_G - U'', \tag{7}$$

where U'' is the second derivative over y. We see from Eq. (6) that the parameter ϵ is a measure of the magnitude of nonlinear products. We should emphasize that the investigation of the problem with the same order of nonlinearity and dispersion is carried out here.

To find the asymptotic solution of the weakly nonlinear problem we will use the multiple scale method and introduce long spatial and temporal time scales $X = \epsilon x$, and $T = \epsilon^3 t$. Thus Eq. (6) now becomes

$$\mathcal{L}_0(\psi) + \epsilon^2\mathcal{L}_1(\psi) + \epsilon J(\psi, \frac{\partial^2\psi}{\partial y^2}) + \epsilon^3 J(\psi, \frac{\partial^2\psi}{\partial X^2}) + \epsilon^4 \frac{\partial^3\psi}{\partial T \partial X^2} = 0. \tag{8}$$

Here the following two linear differential operators are defined as follows

$$\mathcal{L}_0 = [(U - c_0)\frac{\partial^2}{\partial y^2} + p(y)]\frac{\partial}{\partial X}\ , \quad \mathcal{L}_1 = \frac{\partial}{\partial T}\frac{\partial^2}{\partial y^2} + (U - c_0)\frac{\partial^3}{\partial X^3}\ . \tag{9}$$

Now we represent the perturbed stream function ψ as the asymptotic expansion

$$\psi = \psi_0 + \epsilon\psi_1 + \epsilon^2\psi_2 + \dots\ . \tag{10}$$

Substitution of this expression into Eq. (8) allows to get the system of equations and boundary conditions.

1) To the order $O(\epsilon^0)$ we obtain

$$\mathcal{L}_0[\psi_0] = 0, \qquad \psi_0(0) = \psi_0(1) = 0. \tag{11}$$

Suppose $\psi_0 = A(X,T)\Phi_0(y)$. Substitution of this expression into Eqs. (11) gives the following linear differential equation with the appropriate boundary conditions

$$\left(\frac{d^2}{dy^2} + \frac{p(y)}{U - c_0}\right)\Phi_0 = 0, \qquad \Phi_0(0) = \Phi_0(1) = 0. \tag{12}$$

It is assumed that $U - c_0 \neq 0$. By Eqs. (12) an eigenvalue problem for the eigenvalue c_0 is set.

As a result of this subsection we can note: a) we determined the space structure of the wave; b) the system is a time-invariant; c) it is not possible to determine the evolution of the amplitude of the solitary magnetized Rossby waves.

2) To the order $O(\epsilon^1)$, we get

$$\mathcal{L}_0[\psi_1] = -J(\psi_0, \frac{\partial^2\psi_0}{\partial y^2}) \equiv F_1 = A\frac{\partial A}{\partial X}\left(\frac{p(y)}{U - c_0}\right)_y \Phi_0^2 \ , \tag{13}$$

with the boundary conditions $\psi_1(0) = \psi_1(1) = 0$. Here the subscript "$y$" denotes the derivative over y. Further for non-singular neutral modes we represent $\psi_1 = \frac{1}{2}A^2(X,T)\Phi_1(y)$. Substituting this expression into (13) we get

$$\left(\frac{d^2}{dy^2} + \frac{p(y)}{U - c_0}\right)\Phi_1 = \left(\frac{p(y)}{U - c_0}\right)_y \frac{\Phi_0^2}{U - c_0} \ , \tag{14}$$

with the following boundary conditions $\Phi_1(0) = \Phi_1(1) = 0$.

It is clear that this stage of approximation, i.e. $O(\epsilon^1)$ does not give the equation for the amplitude A of the waves, so we should solve a higher order approximation.

3) To the order $O(\epsilon^2)$, we have

$$\mathcal{L}_0[\psi_2] = -\mathcal{L}_1[\psi_0] - J(\psi_0, \frac{\partial^2\psi_1}{\partial y^2}) - J(\psi_1, \frac{\partial^2\psi_0}{\partial y^2}) \equiv F_2 \ , \tag{15}$$

with the boundary conditions $\psi_2(0) = \psi_2(1) = 0$.

Thus at this stage of approximation $O(\epsilon^2)$ there is dispersion effect in longitudinal direction with the necessary nonlinear effect. That is why we call them weak dispersion effect and weakly nonlinear effect.

Now we suppose that ψ_2 has the form $\psi_2 = B(X,T)\Phi_2(y)$. Following the transformations given in [5] we get

$$\int_0^1 dy \frac{F_2}{U - c_0}\Phi_0 = 0 \ . \tag{16}$$

Thus we can conclude that if the perturbation problem (10) has efficient solution, then the secular term F_2 must satisfy Eq. (16), otherwise the amplitude of the wave will be infinite which is meaningless. Substitution of F_2 and ψ_1 into Eq. (16) yields the following modified KdV equation

$$\frac{\partial A}{\partial T} + NA^2\frac{\partial A}{\partial X} + D\frac{\partial^3 A}{\partial X^3} = 0 \,. \tag{17}$$

Here

$$N = \frac{I_2}{I_0}\,, \quad D = -\frac{I_1}{I_0}\,, \tag{18}$$

$$I_0 = \int_0^1 dy\ \Phi_0^2(y)\frac{p(y)}{(U-c_0)^2}\,, \quad I_1 = \int_0^1 dy\ \Phi_0^2(y)\,, \tag{19}$$

$$I_2 = \int_0^1 dy\frac{\Phi_0^2(y)}{U-c_0}\left\{\frac{3}{2}\left(\frac{p(y)}{U-c_0}\right)_y \Phi_1(y) - \frac{1}{2}\,\Phi_0^2(y)\left[\left(\frac{p(y)}{U-c_0}\right)_y \frac{1}{U-c_0}\right]_y\right\}\,. \tag{20}$$

The coefficients N and D are related to the function $\beta_G(y)$ and $U(y)$; as to the functions $\Phi_0(y), \Phi_1(y)$ they can be determined by the solution of the eigenvalue problem Eqs. (12), and (14), respectively.

Discussion. In the given paper nonlinear interaction of magnetized Rossby waves with sheared zonal flow in the Earth's ionospheric E-layer is investigated. It is shown that in case of weak nonlinearity 2D Charney vorticity equation can be reduced to the one-dimensional modified KdV equation. In addition we extended $\beta-$plane approximation of [5] for the arbitrary dependence of Coriolis parameter on the latitude coordinate. Obtained mKdV equation (17) describes the dynamics of amplitude of solitary magnetized Rossby waves and contains the main characteristics of solitary magnetized Rossby waves in a basic flow.

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