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ON TWO CLASSES OF NONLINEAR PARTIAL DIFFERENTIAL SYSTEMS

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Abstract. Two classes of nonlinear partial differential models are considered. The first is based on well known Maxwell's system describing electromagnetic field penetration in the substance. The second is some generalization of the biological model arising in the simulation of vein formation in meristematic tissues of young leaves. Decomposition algorithm with respect to physical processes for the first system and averaged model of sum approximation for the second one are stated. The convergence theorem is also given for the second model.

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Process of electromagnetic field penetration in the substance is described by well known system of Maxwell's equations. In the quasi-stationary approximation this system has the form [1]:

$$\frac{\partial H}{\partial t} = -rot(\nu_m rot H), \tag{1}$$

$$\frac{\partial \theta}{\partial t} = \nu_m \left(rot H\right)^2, \tag{2}$$

where $H = (H_1, H_2, H_3)$ is a vector of the magnetic field, θ is temperature, ν_m characterizes electro-conductivity of the substance. As a rule this coefficient is a function of argument θ. Equations (1) describe the process of diffusion of the magnetic field and equation (2) - change of the temperature at the expense of Joule's heating.

For a more thorough description of electromagnetic field propagation in the medium, it is desirable to take into consideration different physical effects, first of all heat conductivity of the medium has to be taken into consideration. In this case together with (1) instead of (2) the following equation should be considered [1]

$$\frac{\partial \theta}{\partial t} = \nu_m \left(rot H\right)^2 + div\left(\kappa \, grad \, \theta\right), \tag{3}$$

where κ is a coefficient of heat conductivity. As a rule this coefficient is a function of the argument θ as well.

The literature on the questions of existence, uniqueness, regularity, asymptotic behavior of the solutions and numerical resolution of the initial-boundary value problems for (1), (2) and (1), (3) type models in 1D, 2D and 3D cases is very rich (see, for example, [2]-[12] and references therein).

Besides essential nonlinearity, complexities of the mentioned systems (1), (2) and (1), (3) are caused by multi-dimensionality. It is well known that, the general method for construction of economic algorithms for multi-dimensional problems of mathematical physics is a decomposition method. This method allows the reduction of multi-dimensional problems to the set of one-dimensional ones, whose numerical realizations obviously need less computer resources (see, for example, [13] and references therein).

Complex nonlinearity dictates also to split along to the physical processes and investigate basic model by them. In particular, it is logical to split system (1), (3) into the following two models:

$$\frac{\partial H^{(1)}}{\partial t} = -rot\left(\nu_m\left(\theta^{(1)}\right) rotH^{(1)}\right), \quad \frac{\partial \theta^{(1)}}{\partial t} = \nu_m\left(\theta^{(1)}\right)\left(rotH^{(1)}\right)^2, \tag{4}$$

and

$$\frac{\partial H^{(2)}}{\partial t} = -rot\left(\nu_m\left(\theta^{(2)}\right) rotH^{(2)}\right), \quad \frac{\partial \theta^{(2)}}{\partial t} = div\left(\kappa\left(\theta^{(2)}\right) grad\,\theta^{(2)}\right), \tag{5}$$

In (4) Joule's rule, while in (5) process of thermal conductivity are considered.

Investigation of splitting along to the physical processes in one-dimensional case is the natural beginning of studying this issue. In this direction the first step was made in [2]. Some aspects of such splitting are also given in [10], [14]. Various numerical experiments for the studied schemes in one-dimensional case are carried out in [2], [11] and in a number of other works as well. It is very important to continue investigations for 2D and 3D cases of (4), (5).

The second model to be considered is the following multi-dimensional nonlinear system with the first type boundary conditions and initial data:

$$\frac{\partial U}{\partial t} = \sum_{i=1}^{p} \frac{\partial}{\partial x_i}\left(V_i \frac{\partial U}{\partial x_i}\right), \tag{6}$$

$$\frac{\partial V_i}{\partial t} = -V_i + g_i\left(V_i \frac{\partial U}{\partial x_i}\right) + \frac{\partial^2 V_i}{\partial x_i^2}, \quad i = 1, ..., p, \tag{7}$$

$$U(x,t) = 0, \quad (x,\,t) \in \partial\Omega \times [0,T], \tag{8}$$

$$V_i|_{x_i=0} = V_i|_{x_i=1} = 0, \quad t \in [0,\,T], \quad i = 1, ..., p, \tag{9}$$

$$U(x,0) = U_0(x), \quad V_i(x,0) = V_{i0}(x), \quad x \in \overline{\Omega}, \quad i = 1, ..., p. \tag{10}$$

Here $x = (x_1, ..., x_p)$, $\Omega = (0,\,1) \times \cdots \times (0,\,1)$, $\partial\Omega$ is the boundary of the domain Ω, T is some fixed positive number, U_0, V_{i0}, g_i are given sufficiently smooth functions, such that:

$$V_{i0}(x) \geq \delta_0, \quad x \in \overline{\Omega}, \tag{11}$$

$$\gamma_0 \leq g_i(\xi_i) \leq G_0, \quad |g_i'(\xi_i)| \leq G_1, \quad \xi_i \in R, \quad i = 1, ..., p, \tag{12}$$

where δ_0, γ_0, G_0, G_1 are some positive constants.

In the two-dimensional case and without second order derivative terms in (7) by the system (6), (7) the biological model of simulation of vein formation in meristematic tissues of young leaves is described [14]. Many works are dedicated to the investigation and numerical solution of one-dimensional and multi-dimensional analogues of such biological models (see, for example, [14] - [20] and references therein).

Naturally there arises the possibility of reduction of multi-dimensional system (6), (7) to the suitable one-dimensional ones. Let us introduce time step $\tau = T/M$, where M is a natural number and on each segment $[k\tau,\, (k+1)\tau]$, $k = 1, ..., M$, for the initial-boundary value problem (6) - (12) let us consider the following one-dimensional averaged model of sum approximation:

$$\eta_i \frac{\partial u_i^k}{\partial t} = \frac{\partial}{\partial x_i}\left(v_i^k \frac{\partial u_i^k}{\partial x_i}\right), \quad \frac{\partial v_i^k}{\partial t} = -v_i^k + g_i\left(v_i^k \frac{\partial u_i^k}{\partial x_i}\right) + \frac{\partial^2 v_i^k}{\partial x_i^2}, \tag{13}$$

$$u_i^k\big|_{x_i=0} = u_i^k\big|_{x_i=1} = 0, \quad v_i^k\big|_{x_i=0} = v_i^k\big|_{x_i=1} = 0, \tag{14}$$

$$u_i^0(x,0) = U_0(x), \quad v_i^0(x,0) = V_{i,0}(x), \tag{15}$$

$$u_i^k(x,t_k) = u^{k-1}(x,t_k), \quad v_i^k(x,t_k) = v_i^{k-1}(x,t_k), \tag{16}$$

$$u^k(x,t) = \sum_{i=1}^{p} \eta_i u_i^k(x,t), \quad \eta_i > 0, \quad \sum_{i=1}^{p} \eta_i = 1. \tag{17}$$

The following statement takes place.

Theorem. *If the differential problem (6) - (12) has the sufficiently smooth solution then the solution of the averaged model (13) - (17) converges to the exact solution when $\tau \to 0$ and the following estimate holds*

$$\left\|u^k(t) - U(t)\right\| + \sum_{i=1}^{p} \left\|v_i^k(t) - V_i(t)\right\| = O\left(\tau^{1/2}\right).$$

Here $\|\cdot\|$ is the usual L_2 norm.

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