Reports of Enlarged Session of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 29, 2015

ON A REPRESENTATION OF THE SOLUTIONS OF A SECOND-ORDER DIFFERENTIAL EQUATIONS WITH RANDOM COEFFICIENTS

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Abstract. A representation of the solution of second-order ordinal differential equation with random coefficients and random right side is given.

Keywords and phrases: Green's function, generalised stochastic integral, representation formula.

AMS subject classification: 60H10, 34BD5.

Suppose $\{\Omega, \Im, P\}$ is a probability space; $L_2 = L_2([0,1] \times \Omega)$ is a space of real-valued functions having finite second-order moments $\|x\|^2 = E\int_0^1 x^2(t)dt < \infty$. The scalar product in this space has the form $(x,y) = E\int_0^1 x(t)y(t)dt$. Suppose $w(t)$ is the Wiener process.
Consider the boundary value problem

$$y''(t) + \alpha(t)y(t) = w'(t), \tag{1}$$

$$y'(0) = y'(1) = 0, \quad (modP), \quad t \in [0,\, 1].$$

where $\alpha(t)$ satisfies the following conditions

$$\alpha(t) = a(t) + \int_0^1 A(t,s)dw(s). \tag{2}$$

$a(t)$ and $A(t,s)$ are non-random functions $a(t) \in L_2([0,1])$, and $A(t,s) \in L_2([0,1]^2)$. Then $\alpha(t) \in L_2([0,1] \times \Omega)$ and represents a Gaussian process for which $E\alpha(t) = a(t)$ and the correlation kernel has the form

$$K(t,s) = a(t)a(s) + \int_0^1 A(t,\tau)A(s,\tau)d\tau.$$

Consider the problem equivalent to (1):

$$y'(t) + \int_0^t \alpha(s)y(s)ds = w(t),$$

$$\int_0^1 \alpha(s)y(s)ds = w(1), \quad (modP).$$

Our aim is to construct the solution of (1). Therefore, we consider the direct and inverse problems

$$\begin{aligned} &y_1''(t) + \alpha(t)y_1(t) = 0, \\ &y_1(0) = 1,\ y_1'(0) = 0, \end{aligned} \tag{3}$$

$$\begin{aligned} &y_2''(t) + \alpha(t) y_2(t) = 0, \\ &y_2(1) = 1, \ y_2'(1) = 0. \end{aligned} \tag{4}$$

The above-listed properties of the function $\alpha(t)$ satisfy the existence and uniqueness conditions of the solutions of (3), (4).

The Wronskian of this system is $V(t) = y_1(1) \neq 0$. Hence the system y_1, y_2 is independent.

Construct Green's function for this problem

$$G(t,s) = \begin{cases} y_1(t) y_2(s) V^{-1}(0), & \text{t} \leq \text{s}, \\ y_1(s) y_2(t) V^{-1}(0), & \text{t} > \text{s}. \end{cases} \tag{5}$$

Let us try to write the solution of problem (1) in the Daletsky-Skorokhod extended stochastic integral of the form

$$y(t) = \int_0^1 \langle G(t,s), dw(s) \rangle . \tag{6}$$

First we have to show the existence of this integral. Consider the expression

$$E \int_0^1 \left(\frac{\delta G(t,s)}{\delta w(u)} \right)^2 du.$$

Note that $\dfrac{\delta \alpha(t)}{\delta w(u)} = A(t,u)$ and

$$\frac{\delta y_1(t)}{\delta w(u)} = \sum_{k=1}^{\infty} (-1)^k \int_0^t \int_0^{\lambda_1} \int_0^{\lambda_2} ... \int_0^{\lambda_{2k-1}} \sum_{i=1}^{k} A(t_{2i}, u) \prod_{\substack{j=1 \\ j \neq i}}^{k} \alpha(\lambda_{2j}) d\lambda_{2k} \cdot ... \cdot d\lambda_3 d\lambda_2 d\lambda_1,$$

$$\frac{\delta y_2(t)}{\delta w(u)} = \sum_{k=1}^{\infty} (-1)^k \int_t^1 \int_{\lambda_1}^1 \int_{\lambda_2}^1 ... \int_{\lambda_{2k-1}}^1 \sum_{i=1}^{k} A(t_{2i}, u) \prod_{\substack{j=1 \\ j \neq i}}^{k} \alpha(\lambda_{2j}) d\lambda_{2k} \cdot ... \cdot d\lambda_3 d\lambda_2 d\lambda_1.$$

Take into consideration that for Gaussian values we have

$$E \prod_{k=1}^{n} \alpha(t_{2k}) = 0, \text{ for } \ n = 2m - 1, \ \ m = 1, 2, ...,$$

$$E \prod_{k=1}^{n} \alpha(t_{2k}) \leq (2m-1)!! A^m, \text{ for } \ n = 2m, \quad m = 1, 2, ...,$$

where the constant A is selected so that

$$\int_0^1 A(t,s) ds < A.$$

Thus,

$$\int_0^1 E \left[\frac{\delta y_1(t)}{\delta w(u)} y_2(s) V^{-1}(0) \right]^2 du \leq \int_0^1 E \left[\sum_{k=1}^{\infty} \sum_{m=0}^{\infty} (-1)^{k+m} \int_0^t \int_0^{\lambda_1} ... \int_0^{\lambda_{2k-1}} \int_t^1 \cdot \right.$$

$$... \int_0^{\lambda_{2k-1}} \int_t^1 \int_{\tau_1}^1 ... \int_{\tau_{2m-1}}^1 \sum_{i=1}^{k} A(t_{2k}, u) \prod_{\substack{j,m=1 \\ j\neq i}}^{k} \alpha(\lambda_{2j})\alpha(\tau_{2m})d\tau_{2m}...d\tau_1 d\lambda_{2j}...d\lambda_1 \Bigg]^2 du$$

$$\leq \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} \frac{(2m-1)!!A^m}{\prod_{i=1}^{m}(2i+1)!} < \infty.$$

Similarly, for the second component we have

$$\int_0^1 E\left[y_1(t)\frac{\delta y_2(s)}{\delta w(u)}V^{-1}(0)\right]^2 du < \infty.$$

These estimates show that there exist necessary moments of a stochastic derivative, which ensures the existence of the Daletsky-Skorokhod (6) stochastic integral. Thus we have proved the validity of the following statement.

Theorem 1. *If for boundary value problem (1) Condition (2) is satisfied, where $a(t) \in L_2([0,1])$ and $A(t,s) \in L_2([0,1]^2)$ then Problem (1) has a unique solution with probability 1, which will be given in a stochastic integral form (6), where Green's function $G(t,s)$ is defined by Formula (5) using initial Problems (3) and (4).*

Suppose we have the second boundary value problem for the following equation

$$y''(t) + a(t)y(t) = w'(t) + f'(t), \tag{7}$$

$$y'(0) = y'(1) = 0, \quad (modP), \quad t \in [0,1],$$

where $f(t) \in W^1([0,1])$ and $a(t)$ are determined continuous functions. By virtue of Theorem 1 the solution of this problem can be written in the form of a usual stochastic integral

$$y(t) = \int_0^1 G(t,s)dw(t) + \int_0^1 G(t,s)df(s).$$

Together with problem (7) consider the problem:

$$x''(t) + a(t)x(t) = w(t), \tag{8}$$

$$x'(0) = x'(1) = 0, \quad (modP), \quad t \in [0,1].$$

It is evident that for these two processes we have the equality

$$y(t) = x(t) + \int_0^1 G(t,s)dw_2(t). \tag{9}$$

Suppose μ_y and μ_x , respectively, are distributions $y(t)$ and $x(t)$ of stochastic processes. We are interested in probability continuity of these measures with respect to each other.

Note that measures μ_y and μ_x are real-valued Gaussian measures in a Lebesgue measure space $L_2([0,1])$ of square integrable functions and the mean value of μ_y is

$$Ey(t) = E\left[\int_0^1 G(t,s)dw(s) + \int_0^1 G(t,s)df(s)\right] = \int_0^1 G(t,s)df(s),$$

while the correlation kernel is given by the formula

$$R_y(t,s)=Ey(t)y(s)=E\left[\int_0^1 G(t,\tau)dw(\tau)+\int_0^1 G(t,\tau)df(\tau)\right]\cdot\left[\int_0^1 G(s,\tau)dw(\tau)\right.$$

$$\left.+\int_0^1 G(s,\tau)df(\tau)\right]=\int_0^1 G(t,\tau)G(s,\tau)d\tau+\left[\int_0^1 G(t,\tau)df(\tau)\right]^2.$$

In a similar way we can calculate the characteristics of the measure μ_x:

$$Ex(t)=0, R_x(t,s)=\int_0^1 G(t,\tau)G(s,\tau)d\tau.$$

According to Equalities (9) the equivalence of these measure requires that the necessary and sufficient condition

$$b(t)=\int_0^1 G(t,s)df(s)\in K_xL_2([0,1]), \tag{10}$$

should be satisfied, where K_x is the integral operator $(K_x\varphi)(t)=\int_0^1 K_x(t,s)\varphi(s)ds$, $\varphi=\varphi(t)\in L_2([0,1])$, while the kernel $K_x(t,s)$ is defined by the equality

$$R_x(t,s)=\int_0^1 K_x(t,\tau)K_x(s,\tau)d\tau.$$

It can be easily seen that in our case $K_x(t,s)=G(t,s)$ and Condition (10) leads to

$$\int_0^1 G(t,s)r(s)ds=\int_0^1 G(t,s)df(s)$$

the existence of a solution of the integral equation with respect to the function $r(t)$. This solution exists if $f(t)$ is an absolutely continuous function. Hence the following statement is true.

Theorem 2. *If the measures μ_y and μ_x represent, respectively, distributions of the solutions of Problems* (7) *and* (8) *in the space $L_2([0,1])$ where $a(t)$ is a continuous function while $f(t)$ is an absolutely continuous function, then these measures are equivalent and the Radon-Nikodym density has the following form:*

$$\frac{d\mu_y}{d\mu_x}(u)=\exp\left\{-\int_0^1\int_0^1 G(t,s)u(t)df(s)dw(t)-\frac{1}{2}\int_0^1\left[\int_0^1 G(t,s)df(s)\right]^2 dt\right\}.$$

Received 21.05.2015; revised 25.09.2015; accepted 30.10.2015.

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