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ON THE CONVERGENCE OF FOURIER SERIES OF FUNCTIONS OF BOUNDED VARIATION WITH RESPECT TO GENERAL ORTHONORMAL SYSTEMS

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Abstract. S. Banach has proved that for any $f \in L_2([0,1])$, $f \not\sim 0$, there exists ONS $(\psi_n(x))$ such that

$$\overline{\lim_{n\to\infty}} \left| \sum_{k=1}^{n} \widehat{\psi}_k(f)\psi_k(x) \right| = +\infty \quad \text{a.e. on } [0,1].$$

From S. Banach's result it follows that good differential properties of the function do not guarantee the almost everywhere convergence on $[0,1]$ of the series of this function with respect to general ONS.

On the other hand, with respect to classical ONS (the trigonometric system, the Walsh system,...) the functions, having good properties, have the convergent Fourier series.

The problem arises: to single out ONS with respect to which the Fourier series of these functions from $V([0,1])$ are a.e. convergent on $[0,1]$. In this case we should impose some conditions on the functions of ONS $(\varphi_n(x))$.

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Assume that $(\varphi_n(x))$ is an orthonormal system (ONS) on $[0,1]$, $f \in L([0,1])$,

$$\widehat{\varphi}_n(f) = \int_0^1 f(x)\varphi_n(x)\,dx, \quad n = 1, 2, \dots ,$$

are Fourier coefficients of the function f with respect to the system $(\varphi_n(x))$,

$$L_n(x) = \int_0^1 \left| \sum_{k=1}^{n} \varphi_k(x)\varphi_k(t) \right| dt$$

is the Lebesgue function of ONS $(\varphi_n(x))$. As usual, $V([0,1])$ denotes the class of functions of bounded variation.

Introduce the following notation:

$$S_n(a) = \max_{1\le i\le n} \left| \int_0^{\frac{i}{n}} p_n(a,x)\,dx \right|,$$

where

$$p(a,x) = \sum_{k=1}^{n} a_k \lambda_k \varphi_k(x),$$

(a_k) and (λ_k) are certain sequences of numbers.

The following theorem by S. Kaczmarz is well known (see [1, p. 160]).

Theorem (Kaczmarz). *Let $(\varphi_n(x))$ be ONS on $[0,1]$, let λ_n be an increasing sequence of numbers and*

$$\lim_{n\to\infty} \lambda_n = +\infty.$$

If $L_n(x) = O_x(\lambda_n)$ and

$$\sum_{n=1}^{\infty} a_n^2 \lambda_n < +\infty,$$

then the series

$$\sum_{n=1}^{\infty} a_n \varphi_n(x)$$

is convergent a.e. on $[0,1]$.

S. Banach has proved that for any $f \in L_2([0,1])$, $f \not\sim 0$, there exists ONS $(\psi_n(x))$ such that

$$\overline{\lim_{n\to\infty}} \left| \sum_{k=1}^{n} \widehat{\psi}_k(f) \psi_k(x) \right| = +\infty \quad \text{a.e. on } [0,1].$$

From S. Banach's result it follows that good differential properties of the function do not guarantee the almost everywhere convergence on $[0,1]$ of the series of this function with respect to general ONS.

On the other hand, with respect to classical ONS (the trigonometric system (see [1, p. 26]), the Walsh system (see [4])), the functions, having good properties, have the convergent Fourier series.

The problem arises: to single out ONS with respect to which the Fourier series of these functions from $V([0,1])$ are a.e. convergent on $[0,1]$. In this case we should impose some conditions on the functions of ONS $(\varphi_n(x))$.

Definition. Assume that ONS $(\varphi_n(x))$ has the property $D(\lambda)$ if the following conditions

$$L_n(x) = O_x(\lambda_n),$$

$\lambda_n \uparrow \infty$ and

$$\lambda_n = O(\log^2 n)$$

are fulfilled.

Theorem 1. *Let ONS $(\varphi_n(x))$ have the property $D(\lambda)$. If for any sequence $a = (a_n) \in \ell_2$ the following condition (see [1])*

$$S_n(a) = O_a(1) \left(\sum_{k=1}^{n} a_k^2 \lambda_k \right)^{\frac{1}{2}} \tag{1}$$

is satisfied, then the coefficients of the function $f \in V([0,1])$ will satisfy the following condition

$$\sum_{n=1}^{\infty} \widehat{\varphi}_n^2(f) \lambda_n < \infty. \tag{2}$$

From Theorem 1 and Kačmaž theorem we get

Corollary 1. If the conditions of Theorem 1 are satisfied, then Fourier series of any function $f \in V([0,1]$ will almost everywhere converge on $[0,1]$.

Theorem 2. *Let the ONS $(\varphi_n(x))$ have the property $D(\lambda)$. If for some sequence $(b_n) \in \ell_2$ the condition*

$$\overline{\lim_{n\to\infty}} \frac{S_n(b)}{\sum\limits_{k=1}^{n} b_k^2 \lambda_k} = \infty$$

is fulfilled, then we can find the function $f_0 \in V([0,1]$ such that

$$\sum_{n=1}^{\infty} \widehat{\varphi}_n^2(f_0)\lambda_k = +\infty.$$

Corollary 2. From Theorems 1 and 2 there follows the equivalence of conditions (1) and (2).

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