Reports of Enlarged Session of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 29, 2015

ESTIMATES OF FOURIER COEFFICIENTS BY MODULI OF CONTINUITY

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Abstract. It is well known (see [1]) that if $f(x) \in L_2(0,1)$ is an arbitrary function, then its Fourier coefficients with respect to general orthonormal systems (ONS) may be arbitrary from class ℓ_2. Thus it is impossible to estimate these coefficients by moduli of continuity or moduli of smoothness of the given function.

In the present paper the conditions will be given which should be satisfied by the functions of ONS $(\varphi_n(x))$, so that they provide the estimates of Fourier coefficients from the given classes of functions by moduli of smoothness.

Keywords and phrases: Fourier coefficients, moduli of continuity.

AMS subject classification: 42A20.

Denote, as usual, by $L_p(0,1)$ $(p>0)$ the class of all functions for which

$$\int_0^1 |f(x)|^p dx < +\infty.$$

Denote by $C(0,1)$ the class of all functions, continuous on $[0,1]$.
Let $f(x) \in C(0,1)$. Then

$$\omega^{(2)}(\delta, f) = \sup_{|x-y|\le\delta} \left| f(x) - 2f\left(\frac{x+y}{2}\right) + f(y) \right|$$

is the smoothness modulus of the function $f(x)$.
The integral modulus of continuity of functions $f(x) \in L_p(0,1)$ has the form

$$\omega_p(\delta, f) = \sup_{|h|\le\delta} \left(\int_0^{1-h} |f(x+h) - f(x)|^p dx \right)^{\frac{1}{p}}.$$

Let $(\varphi_n(x))$ be ONS on $[0,1]$. The Fourier coefficients of functions $f(x) \in L_1(0,1)$ are defined as follows

$$\widehat{\varphi}_n(f) = \int_0^1 f(x)\varphi_n(x)\,dx, \quad n = 1, 2, \ldots .$$

For the trigonometric system the following relations (see [2])

$$a_n(f) = O(1)\omega^{(2)}\left(\frac{1}{n}, f\right) \quad \text{and} \quad b_n(f) = O(1)\omega^{(2)}\left(\frac{1}{n}, f\right) \tag{1}$$

are satisfied, where $f(x) \in C(0,1)$.

The similar estimates for the Haar and Walsh system are not known. In the present paper, we will however consider the above-mentioned question.

Let $(\varphi_n(x))$ on $[0,1]$ and $\Phi_n(x) = \int_0^x \varphi_n(t)\,dt$.

Suppose

$$H_n = n\sum_{k=1}^{n-1}\left|\int_0^{\frac{k}{n}} \Phi_n(x)\,dx\right|.$$

We have

Theorem 1. *Let $(\varphi_n(x))$ be ONS on $[0,1]$ such that $\int_0^1 \varphi_n(x)\,dx = 0$ and $\int_0^1 \Phi_n(x)\,dx = 0$, $n = m, m+1, \dots)$.*

If $H_n = O(1)$, then for all functions $f(x)$, which satisfy the condition $f'(x) \in L_2(0,1)$ we have

$$\widehat{\varphi}_n(f) = O(1)\left(\omega^{(2)}\left(\frac{1}{n}, f\right) + \frac{1}{\sqrt{n}}\,\omega_2\left(\frac{1}{n}, f'\right)\right).$$

There arises the question with respect to Theorem 1: what will we have if the condition $H_n = O(1)$ is not satisfied?

The answer to this question is given by

Theorem 2. *Let $(\varphi_n(x))$ be ONS on $[0,1]$ and $\int_0^1 \varphi_n(x)\,dx = 0$, $\int_0^1 \Phi_n(x)\,dx = 0$, $(n = m, m+1, \dots)$.*

If $\overline{\lim\limits_{n\to\infty}} H_n = +\infty$, then there exists the function $f_0(x) \in C(0,1)$ such that

$$\overline{\lim_{n\to\infty}} \frac{|\widehat{\varphi}_n(f_0)|}{\omega^{(2)}(\frac{1}{n}, f_0)} = +\infty.$$

One more question arises here: is the condition $\int_0^1 \Phi_n(x)\,dx = 0$ necessary in Theorem 1?

Theorem 3. *If ONS $(\varphi_n(x))$ satisfies the conditions*

a) $\int_0^1 \varphi_n(x)\,dx = 0, \quad n = m, m+1, \dots,$

b) $\widehat{\varphi}_n(f) = O(1)\omega^{(2)}\left(\frac{1}{n}, f\right)$ *for any* $f(x) \in C(0,1)$,

then $\int_0^1 \Phi_n(x)\,dx = 0$, $n = m, m+1, \dots$.

And another question arises: are the conditions $\int_0^1 \varphi_n(x)\,dx = 0$ and $\int_0^1 \Phi_n(x)\,dx = 0$, $(n = m, m+1, \dots)$ sufficient for the estimate $\widehat{\varphi}_n(f) = O(1)\omega^{(2)}\left(\frac{1}{n}, f\right)$ $(f(x) \in C(0,1))$?

Theorem 4. *There exists ONS $(\varphi_n(x))$ on $[0,1]$ such that the conditions*

1) $\int_0^1 \varphi_n(x)\,dx = 0, \quad n = 1, 2, \dots,$

2) $\int_0^1 \Phi_n(x)\,dx = 0, \quad n = 1, 2, \dots$

are simultaneously satisfied. For the functions $f_0(x) = x^2$, however, the condition

$$\widehat{\varphi}_n(f_0) = O(1)\omega^{(2)}\left(\frac{1}{n}, f_0\right)$$

is not satisfied.

From Theorems 2 and 3 it follows that for the Haar and Walsh systems the condition $H_n = O(1)$ or $\int_0^1 \Phi_n(x)\,dx = 0$, $(n = m, m+1, \dots)$ is not satisfied.

Efficiency of condition of Theorem 1

Theorem 5. *For the trigonometric system the estimate $H_n = O(1)$ is valid.*

Theorem 6. *For the Haar (see [2, p. 70]) and Walsh (see [4]) systems, the condition $H_n = O(1)$ is not satisfied.*

Thus the Fourier coefficients of continuous functions with respect to Haar and Walsh systems are not estimated by smoothness moduli of these functions.

Acknowledgement: The authors thank Shota Rustaveli National Scientific Foundation (grant No. FR/223/5-100/13) for the financial support.

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Received 11.05.2015; revised 20.11.2015; accepted 11.12.2015.

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