Reports of Enlarged Session of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 29, 2015

# SIMULATION OF BURRIDGE-KNOPOFF MODEL

Chargazia Kh., Kharshiladze O.

**Abstract**. Simulation of Burridge-Knopoff spring-block model is carried out. Discrete equations are used for this purpose. The discrete system exhibits both periodic and chaotic motion, the system's transition to chaos is size-dependent (dependent on blocks' number). The Dietrich -Ruina friction law attached to the model is nonlinear and scale-dependent, which represents an implication of the numerical simulation. The critical parameter value needed for the onset of chaos in a single block case is defined.

**Keywords and phrases**: Friction, numerical solutions, earthquake dynamics, chaos.

**AMS subject classification**: 35J05, 35A24, 39A70.

Dynamic model, studied extensively since its introduction in the 1960s, is the Burridge and Knopoff [1967, 1] (BK) model of many blocks interconnected by elastic springs with constant spring stiffness coefficient. The blocks are also elastically coupled (with spring stiffness coefficient to a rigid plate moving at a constant velocity $V_0$ and pulled over a rough surface described by some friction law. The interface between the blocks and the rough surface can be considered an analogue for a 1-dimensional earthquake fault [Carlson et al., 1991]. Burridge and Knopoff [1967, 1] conducted several laboratory experiments of this system - the first case considered equal spring constants between blocks, and the second with graduated values for the spring constants. They observed several types of behavior in this configuration including the presence of large shocks in the system when the spring constants were stretched far enough to set the blocks on the verge of instability. And while they found a Gutenberg-Richter distribution of event sizes present in their model, they note that statistical properties along the fault surface are determined by the nature of the friction law describing the interface (a property confirmed, at least partially, by Elbanna and Heaton [2009, 3]). At this time rate and state friction laws had not yet emerged as powerful tools in dynamic simulations however. Burridge and Knopoff formulated the equations of motion for this system incorporating a friction law that was dependent only on the block's velocity.

The equations of motion are derived from a one-dimensional chain of spring-connected blocks elastically coupled and driven by a plate moving at a constant rate $V_0$ according to the model by using the version proposed by Madariaga [1998, 2, 6] of a single spring-block oscillator. The blocks slide along a rough surface according to the nonlinear D-R friction law and the equations of motion for the j-th block's position $u_j$ are given by

$$mu_j = \mu(u_{j+1} - 2u_j + u_{j-1}) - \lambda u_j - F_j(u_j, \theta_j), \tag{1}$$

$$F_j(u_j, \theta_j) = \theta_j + A\ln(u_j/V_0),\ \theta_j = -((u_j + V_0)/D_c)(\theta_j + B\ln(u_j/V_0 + 1)),$$

where $F_j$ is the rate $u_j$ and state $\theta_j$ Dietrich - Ruina friction law from equation, $\mu$ is the spring constant coupling the blocks, $\lambda$ is the spring constant coupling each block to the driver plate, and $V_0, A, B$ and $D_c$ are the constant frictional parameters, $u_j$ is the position of the j-th block, or its slip from its initial starting position.

Because of the nonlinearity imposed into equation (1) by the logarithmic term in the D-R friction law, analytic integration cannot be done even in the simplest case of a single block. For this reason we proceed by implementing a numerical method by first writing (1) as a system of three first order ODEs

$$u_j = v_j, \ \ v_j = \gamma^2(u_{j+1} - 2u_j + u_{j-1}) - \tilde{\gamma}^2 u_j - (\gamma^2/\xi)(\theta_j + ln(u_j/V_0 + 1)), \tag{2}$$

$$\theta_j = -(v_j + 1)(\theta_j + (1 + e)ln(u_j/V_0 + 1)),$$

where $\gamma = \sqrt{\mu/m}(D_c/V_0)$,$\tilde{\gamma}\sqrt{\lambda/m}(D_c/V_0)$, are non-dimensional frequencies, $\xi(\mu D_c)/A$ is a non-dimensional spring constant, $e = (B - A)/A$.

Dieterich-Ruina friction has introduced numerical challenges because the nonlinearity of the logarithmic term causes the system's local Jacobian matrix to have very negative eigenvalues - a property that usually indicates the presence of numerical stiffness (well documented in [Erickson et al., 2008, 4] and [Noda et al., 2008, 7]). During the simulations it was found that even with the use of an implicit numerical method suited for numerically stiff problems, the time step was still restricted by accuracy requirements. Even with a stable method, if the time step taken is too large, then the algorithm returns numerical value of $v_j < -1$ and the logarithmic term is undefined. For this reason, we use a classical fourth order explicit Runge-Kutta method on the ODEs in equation (2) whose step size adapts according to requirements for accurate resolution when $v_j$ is close to -1. N blocks are evenly spaced on a chain of length 20 dimensionless spatial units. Since fault rupture is caused by small stress instabilities along the fault surface and often propagate like a localized pulse [Heaton, 1990, 3,5], we choose to represent the initial data as localized departure from the equilibrium (or stationary) regime. Therefore initial data is a smooth Gaussian pulse centered at the middle block

$$u_0(j) = 1.5e^{(x_i - 10)^2/\sigma^2}, \ \ v_0(j) = 0, \ \ \theta_0(j) = 0, \ \ j = 1, 2, \ldots, N, \tag{3}$$

where $\sigma = 1$.

This corresponds to imposing an initial stress peturbation into the initial position of each block from its adjacent point on the driver plate, the middle block having the greatest initial displacement. All have zero initial velocity (with respect to the driver plate). Free boundary conditions imply that blocks on either end of the chain are only influenced by the single block connecting them to the chain, and their elastic coupling with the driver plate. Because of a lack of insight into proper parameter values, we explore the parameter space that allows for more manageable numerical computation (i.e. where the parameters associated with the nonlinear terms are not too large). Numerical integration is done for different amounts of blocks: $N = 1, 10, 20, 21$ blocks. Parameter values used here are fixed at $e = 0.5; \xi = 0.5; \gamma = 0.5; \tilde{\gamma} = 0.5$.. Figures 1 - 3 correspond to three blocks. Figure 1 is the initial displacement of all N blocks and figure 2 is the slip of all N blocks against time. Figure 3 is the contour of the middle block's slip against time and one can further view the periodic, or aperiodic behavior occurring. Periodic orbits will appear as a single closed loop in the phase space, while aperiodic orbits will appear as a strange attractor.

After a transient period in which the initial perturbation is amplified, the nonlinearities saturate this growth and the system settles into the same periodic trajectory - suggesting that the blocks move collectively. All 3 blocks undergo abrupt, periodic motion of period approximately 20 temporal units and amplitude approximately 4 slip units. They are stuck to the rough surface until the driver plate overcomes the static friction holding each block in place, and the chain suddenly begins to slide. After a time period, the blocks approach a gradual stop until the pulling force overcomes static friction, and the cycle begins again. Sudden motion, reminiscent of stick-slip behavior emerges as the blocks respond to the driver plate. Under the same parameter combination, periodic motion occurs when considering the system of 10 blocks as viewed in although it appears that the period of the solution has undergone at least one period doubling bifurcation. In this case all 10 blocks undergo periodic motion, but their slip values reach different amplitudes - the block in the center of the chain reaches an amplitude of approximately 3 slip units during each cycle, while nearby blocks reach smaller amplitudes. Similar is the case for N= 20 blocks, although the periodicity of the motion appears to double again. For this fixed set of parameter values, the motion is periodic in time for N = 3,...,20, but when N = 21, the motion becomes aperiodic in time. Each block follows its own chaotic trajectory in time and the blocks appear to move independently of each other suggesting chaotic behavior in space as well. Further studies show that this transition to chaotic motion varies, depending on the parameters considered. More specifically, the fact that a transition occurs between N = 20 and N = 21 is not universal; it depends on the parameters used. It is also important to note that regardless of the type of motion these systems produce, one can observe that there is a transient period during which small instabilities introduced by the initial slip displacement is amplified as energy enters the system. This amplification is then saturated by the nonlinearities present from the friction law. This feature suggests that under these parameter values, the friction law can be a mechanism responsible for causing even small instabilities to grow into large, but finite events.

**Conclusion**. We have made the simulations of the discrete formulations of a one-dimensional Burridge and Knopoff [1967, 1] spring-block model subject to the nonlinear Dieterich-Ruina friction law. In the discrete case we observe a transition to chaos when varying the system size, i.e. the number of blocks N. For N = 3, 10, and 20 blocks, periodic behavior emerges. When N is increased from 20 to 21 however, this periodic behavior is lost and chaos ensues, as further asserted by the broadband noise in the power spectrum. This transition occurs for a fixed set of parameter values and we see that the small value of $e = 0,5$ will generate chaotic motion, as long as the system size N is sufficiently large. This value is much smaller than that required for chaotic motion that we found in the single block case, where $e \approx 11$. This suggests that, dynamic rupture modeling with this friction law can produce chaotic dynamics when considering a wide range of parameter values with an increase in system size. It is possible that the parameter values selected by the authors, or their regularization of the D-R friction law prohibited the emergence of chaotic dynamics. Also, these results suggest that chaotic regimes in the BK model under D-R friction is a function of the number of blocks considered, similar to the conclusions of Schmittbuhl et al. [1993, 8]. It should be emphasized that this information reveals that the D-R friction law may very well be scale-dependent, as we have seen different dynamics emerge in systems with different numbers of blocks. That the transition to chaos appears highly sensitive to the number of blocks N as well as the value of the parameter suggests that one should take into consideration their system size when choosing the parameters for a dynamic rupture model, or find another means of scaling the friction law appropriately.

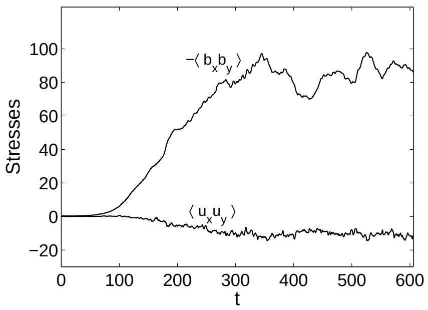


Fig.1.

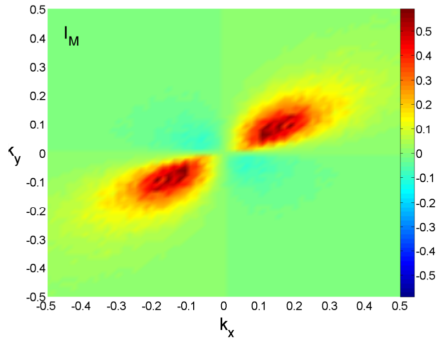


Fig.2.

**Acknowledgment**. The proposed work was supported by Shota Rustaveli Georgian National Science Foundation, Grant $\#N31/14$.

## R E F E R E N C E S

1. Burridge R., Knopoff L. Model and theoretical seismicity. *Bull. Seism. Soc. Am.*, **57** (1967), 341-371.

2. Carlson J.M., Langer J.S, et all. Intrinsic properties of a burridge-knopoff model of an earthquake fault. *Phys. Rev. A.*, **44**, 2 (1991), 884-897.

3. Elbanna A., Heaton T. Statistics of a dynamical system failing at multiple length scales and its implications on material strength. *Poster presented at the Southern California Earthquake Center (SCEC) annual meeting, Abstract published in Proceedings and Abstracts*, **19** (2009).

4. Erickson B., Birnir B., Lavalle D. A model for aperiodicity in earthquakes.*Nonlin. Processes Geophys.*, **15** (2008), 1-12.

5. Heaton T.V. Evidence for and implications of self-healing pulses of slip in earthquake rupture. *Phys. Earth. Planet. Inter.*, **64** (1990), 1-20.

6. Madariaga R. Study of an oscillator of single degree of freedom with dieterich-ruina rate and state friction. Unpublished Notes (Preprint), (1998).

7. Noda H., Dunham E., Rice J. Earthquake ruptures with thermal weakening and the operations of major faults at low overall stress levels. *J. Geophy. Res.,* **B07302** (2008), 1-14.

8. Schmittbuhl J., Vilotte J., Roux S. Propagative macrodislocation modes in an earthquake fault model. *Europhys. Lett.,* **21**, 3 (1993), 375-380.

Received 27.05.2015; revised 14.12.2015; accepted 25.12.2015.

Authors' addresses:

Kh. Chargazia, O. Kharshiladze
I. Vekua Institute of Applied Mathematics and
M. Nodia Institute of Geophysics of
Iv. Javakhishvili Tbilisi State University
2 University St., Tbilisi 0186 Georgia
E-mail: khatuna.chargazia@gmail.com