Reports of Enlarged Session of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 29, 2015

ON THE FERNIQUE-SKOROKHOD TYPE INTEGRAL

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Abstract. The Fernique-Skorokhod type integral is computed from the exponential of the sum of linear and quadratic functionals.

Keywords and phrases: Hilbert space, Gaussian random variable, Fernique-Skorokhod type integral.

AMS subject classification: 47B80, 60H05.

Let H be a separable, real Hilbert space with the scalar product $(\cdot,\cdot)$ and the norm $\|\cdot\|$; ξ is the Gaussian random variable with values in H. Furthermore, ξ has a kernel (linear, kernel type and positively defined operator) correlation operator B and $E\xi = 0$. Suppose, that $a \in H$. Let R be a kernel type operator too. The mathematical expectations

$$E\exp\{(a,\xi)_H + (R\xi,\xi)_H\} \tag{1}$$

are the Fernique-Skorokhod type integrals (see [1,2]). Our aim is to prove the formula:

$$\begin{aligned} &E\exp\{(a,\xi)_H + (R\xi,\xi)_H\} \\ &= \det\left(I - 2B^{\frac{1}{2}}RB^{\frac{1}{2}}\right)^{-\frac{1}{2}} \cdot \exp\left\{\tfrac{1}{2}\left(B^{\frac{1}{2}}\left(I - 2B^{\frac{1}{2}}RB^{\frac{1}{2}}\right)^{-1}B^{\frac{1}{2}}a, a\right)_H\right\}. \end{aligned} \tag{2}$$

Earlier version of this formula was discussed in [3,4].

Let $\{e_k\}$ be the orthonormal system of eigenvectors of the operator $B^{\frac{1}{2}}RB^{\frac{1}{2}}$ and $\{\lambda_k\}$ are eigenvalues related to vectors $\{e_k\}$. The scalar product $(x,y) = \left(B^{\frac{1}{2}}x, B^{\frac{1}{2}}y\right)$, $x \in H$, $y \in H$ in H is introduced, and closure of H on norm $\|x\|_- = \sqrt{(x,x)}$ is considered. Denote the obtained space by H_-. Further, H_+ is the subspace of space H and range of definition of the operator $B^{-\frac{1}{2}}$. H_+ is the Hilbert space in scalar product $(x,y)_+ = \left(B^{\frac{1}{2}}x, B^{\frac{1}{2}}y\right)_H$. Thus $H_-^* = H_+$ and the obtained three $H_+ \subset H \subset H_-$ is named as an equipped Hilbert space.

Suppose $\xi = B^{\frac{1}{2}}\zeta$, where ζ is the so called "white noise". It is a random element in space H_- with zero mean and identity correlation operator in H. It is possible to give usual sense as extension on a continuity of a functional to the equality

$$(a,\xi)_H + (R\xi,\xi)_H = \left(B^{\frac{1}{2}}a, \zeta\right)_H + \left(B^{\frac{1}{2}}RB^{\frac{1}{2}}\zeta, \zeta\right)_H.$$

Really, the scalar product $(x,\zeta)_H$ is well defined at $x \in H_+$ and $\zeta \in H_-$. Thus, the possibility of extension of functional $(x,\zeta)_H$ follows from equality $E(x,\zeta)_H^2 = \|x\|_H^2$ in case when $x \in H_-$ (so called measurable random functional). Moreover, $\left(B^{\frac{1}{2}}a, \zeta\right)_H = \sum_{k=1}^{\infty} a_k \zeta_k$, where $a_k = \left(e_k, B^{\frac{1}{2}}a\right)_H$ and $\zeta_k = (e_k, \zeta)_H$. Here ζ_k are independent Gaussian random variables with the parameters equal to 0 and 1. Analogously, we can give sense to the second term. Let P_n be the projection operator on subspace generated by the vectors $e_1, e_2, ..., e_n$. Then

$$\begin{aligned}
(R\xi, \xi)_H &= \left(B^{\frac{1}{2}} R B^{\frac{1}{2}} \zeta, \zeta\right)_H = \lim_{n\to\infty} \left(B^{\frac{1}{2}} R B^{\frac{1}{2}} P_n \zeta, P_n \zeta\right)_H \\
&= \lim_{n\to\infty} \sum_{i,j=1}^{n} \left(B^{\frac{1}{2}} R B^{\frac{1}{2}} e_i, e_j\right)_H \zeta_i \zeta_j = \lim_{n\to\infty} \sum_{i,j=1}^{n} (\lambda_i e_i, e_j)_H \zeta_i \zeta_j \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \lambda_i \zeta_i^2 = \sum_{i=1}^{\infty} \lambda_i \zeta_i^2.
\end{aligned}$$

In view of these reasons we can write

$$\begin{aligned}
&E\exp\{(a,\xi)_H + (R\xi,\xi)_H\} = E\exp\left\{\sum_{k=1}^{\infty} a_k\zeta_k + \sum_{k=1}^{\infty} \lambda_k \zeta_k^2\right\} \\
&= \prod_{k=1}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{a_k x + \lambda_k x^2 - \frac{1}{2}x^2} dx = \prod_{k=1}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp[-0.5 q_k (x-d_k)^2 + 0.5 q_k d_k] dx,
\end{aligned}$$

where $q_k = 1 - 2\lambda_k$, $d_k = \frac{a_k}{1-2\lambda_k}$. Denote $y = \sqrt{q_k}(x - d_k)$, $dx = \frac{1}{\sqrt{q_k}} dy$, then it is easy to calculate the obtained integral:

$$E\exp\{(a,\xi)_H + (R\xi,\xi)_H\} = \prod_{k=1}^{\infty} \frac{1}{\sqrt{q_k}} e^{\frac{q_k d_k}{2}} = \exp\left\{\sum_{k=1}^{\infty}\left[\frac{a_k^2}{2(1-2\lambda_k)} - \frac{1}{2}\ln(1-2\lambda_k)\right]\right\}.$$

It isn't difficult to see that

$$\begin{aligned}
&\prod_{k=1}^{\infty} (1-2\lambda_k)^{-\frac{1}{2}} = \left\{\det\left[I - 2B^{\frac{1}{2}} R B^{\frac{1}{2}}\right]\right\}^{-\frac{1}{2}}; \\
&\frac{e_k}{1-2\lambda_k} = \left(I - 2B^{\frac{1}{2}} R B^{\frac{1}{2}}\right)^{-1} e_k; \\
&\sum_{k=1}^{\infty} \frac{a_k^2}{1-2\lambda_k} = \sum_{k=1}^{\infty} \left(B^{\frac{1}{2}}a, \frac{e_k}{1-2\lambda_k}\right)_H \left(B^{\frac{1}{2}}a, e_k\right)_H \\
&= \sum_{k=1}^{\infty} \left((I - 2B^{\frac{1}{2}} R B^{\frac{1}{2}})^{-1} B^{\frac{1}{2}} a, (B^{\frac{1}{2}}a, e_k)_H e_k\right)_H \\
&= \left((I - 2B^{\frac{1}{2}} R B^{\frac{1}{2}})^{-1} B^{\frac{1}{2}} a, \sum_{k=1}^{\infty} (B^{\frac{1}{2}}a, e_k) e_k\right)_H = \left(B^{\frac{1}{2}}(I - 2B^{\frac{1}{2}} R B^{\frac{1}{2}})^{-1} a, a\right)_H.
\end{aligned}$$

Therefore,

$$
\begin{aligned}
Ee^{(a,\xi)_H+(R\xi,\xi)_H} &= \left(\prod_{k=1}^{\infty}\frac{1}{\sqrt{1-2\lambda_k}}\right)e^{\frac{1}{2}\sum_{k=1}^{\infty}\frac{a_k^2}{1-2\lambda_k}} \\
&= \det\left[I-2B^{\frac{1}{2}}RB^{\frac{1}{2}}\right]^{-\frac{1}{2}}\exp\left\{\frac{1}{2}\left(B^{\frac{1}{2}}\left[I-2B^{\frac{1}{2}}RB^{\frac{1}{2}}\right]^{-1}B^{\frac{1}{2}}a,a\right)_H\right\}.
\end{aligned}
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R E F E R E N C E S

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Received 14.05.2015; revised 30.09.2015; accepted 27.11.2015.

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