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THE BOUNDERY VALUE PROBLEM POSED IN AN INFINITE DOMAIN FOR ONE EQUATION OF TRANSFER THEORY

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Abstract. The aim of this paper is to study the boundary value problem for one non-homogeneous linear transport equation in a infinite medium with simple kernel. To this end the method of expansions by the eigenfunctions of the corresponding characteristic equation is used.

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Consider the following non-homogeneous linear transport equation which occurs when investigating many important problems of mathematical physics

$$\mu\frac{\partial\Psi}{\partial x}+\Psi(x,\mu,E)=\frac{c}{2}\int_{E_0}^{E_1}\int_{-1}^{+1}\alpha(E)\alpha(E')\Psi(x,\mu',E')d\mu'dE'+f(x,\mu,E), \tag{1}$$

$$x\in(-\infty,+\infty),\quad \mu\in[-1,+1],\quad E\in[E_0,E_1]$$

with additional conditions:

i_1)
$$\Psi^+(x_0,\mu,E)-\Psi^-(x_0,\mu,E)=\psi(\mu,E);$$

i_2)
$$\lim_{x\to\infty}\Psi(x,\mu,E)=0$$

$$x_0\in(-\infty,+\infty),\quad \mu\in[-1,+1],\quad E\in[E_0,E_1],$$

where $\alpha(E)$ is a continuous function and suppose that

$$\int_{E_0}^{E_1}\alpha^2(E)dE=1,$$

$0<c<1$, the function $f(x,\mu,E)$ is also continuous, vanishes as $\mid x\mid\to\infty$, is differentiable with respect to x, satisfying H^* conditions [1] with respect to μ, the function $\psi(\mu,E)$ is continuous satisfying H^* conditions with respect to μ. We wish to find vanishing as $\mid x\mid\to\infty$, continuous solution of Eq.(1) satisfying H^* conditions with respect to μ and the additional conditions (i_1) and (i_2).

Similar problem was investigated by K. Case [2].

It is known that [3] the general solution of the homogeneous equation

$$\mu\frac{\partial\Psi_0}{\partial x}+\Psi_0(x,\mu,E)=\frac{c}{2}\int_{E_0}^{E_1}\int_{-1}^{+1}\alpha(E)\alpha(E')\Psi_0(x,\mu',E')d\mu'dE', \tag{2}$$

$$x \in (-\infty, +\infty), \quad \mu \in [-1, +1], \quad E \in [E_0, E_1]$$

can be represented in the form

$$\Psi_0(x, \mu, E) = d_- \varphi_-(\mu, E) e^{-(x-x_0)/\nu_0} + d_+ \varphi_+(\mu, E) e^{(x-x_0)/\nu_0}$$
$$+ \int_{E_0}^{E_1} \int_{-1}^{+1} u(\nu, \zeta) \varphi_{\nu,(\zeta)}(\mu, E) e^{-(x-x_0)/\nu},$$

where $d_\pm = const$,

$$\varphi_\pm(\mu, E) = \frac{c}{2} \frac{\nu_0 \alpha(E)}{\nu_0 \mp \mu}$$

are the regular eigenfunctions with the two roots $\pm\nu_0$ eigenvalues which occur from the following equation

$$1 - \frac{c\nu}{2} \ln \frac{1 + 1/\nu}{1 - 1/\nu} = 0$$

ν_0 is real and does not lie between -1 and $+1$. The singular eigenfunctions have the form

$$\varphi_{\nu,(\zeta)}(\mu, E) = \frac{c\nu}{2} \frac{\alpha(E)\alpha(\zeta)}{\nu - \mu} + (\delta(\zeta - E) - \frac{c\nu}{2} \int_{-1}^{+1} \frac{\alpha(E)\alpha(\zeta)}{\nu - \mu'} d\mu') \delta(\nu - \mu). \quad (3)$$

It is also known that the system of eigenfunctions of the characteristic equation

$$(\nu - \mu)\varphi_\nu(\mu, E) = \frac{c\nu}{2} \int_{E_0}^{E_1} \int_{-1}^{+1} \alpha(E)\alpha(E')\varphi_\nu(\mu', E') d\mu' dE'$$

corresponding to the homogeneous equation (2) is complete for functions $\psi(\mu, E)$ which is continuous and satisfying H^* condition with respect to μ. It is to be shown that one can express $\psi(\mu, E)$ in the form

$$\psi(\mu, E) = a_- \varphi_-(\mu, E) + a_+ \varphi_+(\mu, E) + \int_{E_0}^{E_1} \int_{-1}^{+1} u(\nu, \zeta) \varphi_{\nu,(\zeta)}(\mu, E) d\nu d\zeta. \quad (4)$$

The coefficients $a_\pm$ are readily found

$$a_\pm = \frac{1}{N_\pm} \int_{E_0}^{E_1} \int_{-1}^{+1} \mu \varphi_\pm(\mu, E) \psi(\mu, E) d\mu dE, \quad (5)$$

where

$$N_\pm = \int_{E_0}^{E_1} \int_{-1}^{+1} \mu \varphi_\pm^2(\mu, E) d\mu dE$$

the function $u(\nu, \zeta)$ may be defined by formula

$$u(\nu, \zeta) = \int_{E_0}^{E_1} \int_{-1}^{+1} \mu \tilde{\varphi}_{\nu,(\zeta)}(\mu, E) \psi(\mu, E) d\mu dE, \quad (6)$$

here

$$\tilde{\varphi}_{\nu,(\zeta)}(\mu, E) = \varphi_{\nu,(\zeta)}(\mu, E) + \frac{\rho(\nu)}{1 - \rho(\nu)} \int_{E_0}^{E_1} \alpha(\zeta)\alpha(\zeta')\varphi_{\nu,(\zeta')}(\mu, E) d\zeta'$$

and

$$\rho(\nu) = 1 - \lambda^2(\nu) - \frac{\pi^2 c^2 \nu^2}{4}, \quad \lambda(\nu) = 1 - \frac{c\nu}{2} \int_{-1}^{+1} \frac{d\mu}{\nu - \mu}.$$

In view of Eq. (4) for the function $f(x, \mu, E)$ we can write

$$f(x, \mu, E) = b_-(x)\varphi_-(\mu, E) + b_+(x)\varphi_+(\mu, E)$$
$$+ \int_{E_0}^{E_1} \int_{-1}^{+1} b(x, \nu, \zeta)\varphi_{\nu,(\zeta)}(\mu, E) d\nu d\zeta,$$

where the coefficients $b_\pm(x)$ and $b(x, \nu, \zeta)$ are defined similar as $a_\pm$ and $u(\nu, \zeta)$.

By immediate substitution it is established that the function

$$\Psi(x, \mu, E) = \Psi_0(x, \mu, E) + \int_{x_0}^{x} b_-(s)\varphi_-(\mu, E) e^{-(x-x_0)/\nu_0} ds$$
$$+ \int_{x_0}^{x} b_+(s)\varphi_+(\mu, E) e^{(x-x_0)/\nu_0} ds$$
$$+ \int_{x_0}^{x} \int_{E_0}^{E_1} \int_{-1}^{+1} b(s, \nu, \zeta)\varphi_{\nu,(\zeta)}(\mu, E) e^{-(x-x_0)/\nu_0} d\nu d\zeta ds$$

satisfies the equation (1).

It is seen that under the condition $x > x_0$ the function

$$\Psi(x, \mu, E) = a_+\varphi_+(\mu, E) e^{-(x-x_0)/\nu_0}$$
$$+ \int_{E_0}^{E_1} \int_{0}^{+1} u(\nu, \zeta)\varphi_{\nu,(\zeta)}(\mu, E) e^{-(x-x_0)/\nu} d\nu d\zeta + \int_{x_0}^{x} b_-(s)\varphi_-(\mu, E) e^{-(x-x_0)/\nu_0} ds$$
$$+ \int_{x_0}^{x} b_+(s)\varphi_+(\mu, E) e^{(x-x_0)/\nu_0} ds$$
$$+ \int_{x_0}^{x} \int_{E_0}^{E_1} \int_{-1}^{+1} b(s, \nu, \zeta)\varphi_{\nu,(\zeta)}(\mu, E) e^{-(x-x_0)/\nu_0} d\nu d\zeta ds$$

is solution of Eq.(1) vanishes as $x \to \infty$. In just the same way under the condition $x < x_0$ the function

$$\Psi(x, \mu, E) = -a_-\varphi_+(\mu, E) e^{-(x-x_0)/\nu_0} - \int_{E_0}^{E_1} \int_{-1}^{0} u(\nu, \zeta)\varphi_{\nu,(\zeta)}(\mu, E) e^{-(x-x_0)/\nu} d\nu d\zeta$$
$$+ \int_{x_0}^{x} b_-(s)\varphi_-(\mu, E) e^{-(x-x_0)/\nu_0} ds + \int_{x_0}^{x} b_+(s)\varphi_+(\mu, E) e^{(x-x_0)/\nu_0} ds$$

is solution of Eq.(1) vanishes as $x \to -\infty$.

Now it remains to satisfy the condition (i_1). But if the coefficients $a_\pm$ and $u(\nu, \zeta)$ are defined from formulas (5) and (6) then the condition (i_1) will be fulfilled.

R E F E R E N C E S

1. Muskhelishvili N. Singular Integral Equations *P. Noordooff: Groningrn*, 1953.

2. Case K.M., Zweifel P.F. Linear Transport Theory, *University of Michigan, Addison-Wesley Pub.Co*, 1967.

3. Shulaia D.A. Linear Equation of the Multivelocity Transport Theory (Russian). *J. Vich. Math.i Math. Phys. 23,5: 1125-1140*, 1983.

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