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VARIETIES OF EXPONENTIAL MR-GROUPS

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Abstract. In the present paper some problems of the theory of the varieties of exponential groups are considered.

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Let R be an associative ring with identity. Myasnikov and Remeslennikov [1] introduced the new category of exponential R-groups as a natural generalization of the notion of an R-module to the noncommutative case. Below, we recall basic definitions borrowed from [1, 2].

Let $L = \langle \cdot\, , {}^{-1}, e\rangle$ be the group language (signature); here, $\cdot$ denotes the binary operation of multiplication, ${}^{-1}$ denotes the unary operation of inversion, and e is a constant symbol for the identity element of the group.

We enrich the group language to the language $\mathfrak{L}^*_{gr} = \mathfrak{L}_{gr} \cup \{f_\alpha(g) \mid \ \alpha \in \mathsf{R}\}$, where $f_\alpha(g)$ is a unary algebraic operation.

Definition 1. A Lyndon R-group is a set G with operations, $\cdot$, ${}^{-1}$, e and $\{f_\alpha(g) \mid \alpha \in \mathsf{R}\}$ are defined and the following axioms hold:

(i) the group axioms;

(ii) for all $g, h \in G$ and all elements $\alpha, \beta \in \mathsf{R}$,

$$g^1 = g, \quad g^0 = e, \quad e^\alpha = e; \tag{1}$$

$$g^{\alpha+\beta} = g^\alpha \cdot g^\beta, \quad g^{\alpha\beta} = (g^\alpha)^\beta; \tag{2}$$

$$(h^{-1}gh)^\alpha = h^{-1}g^\alpha h. \tag{3}$$

For brevity, in the formulas expressing the axioms, we write $f_\alpha(g)$ instead of g^α for $g \in G$ and $\alpha \in \mathsf{R}$.

Let $\mathfrak{L}_\mathsf{R}$ denote the category of all Lyndon R-groups. Since the axioms given above are universal axioms of the language $\mathfrak{L}^*_{gr}$, it follows that $\mathfrak{L}_\mathsf{R}$ is a variety of algebraic systems in the language $\mathfrak{L}^*_{gr}$; therefore, general theorems of universal algebra allow us to consider the varieties of R-groups, R-homomorphisms, R-isomorphisms, free R-groups, and so on.

MR-exponential groups. There exit Abelian Lyndon R-groups which are not R-modules (see [3], where the structure of a free Abelian R-group was studied in detail). The authors of [1] augmented Lyndon's axioms (quasi-identity):

(MR) $$\forall g, h \in G, \ \ \alpha \in \mathsf{R} \ \ [g,h] = 1 \Longrightarrow (gh)^\alpha = g^\alpha h^\alpha. \tag{4}$$

Definition 2. An MR-group is a group G on which the operations g^α are defined for all $g \in G$ and $\alpha \in \mathsf{R}$ so that axioms (1)–(4) hold.

Let $\mathfrak{M}_\mathsf{R}$ denote the class of all R-exponential groups with axioms (1)–(4). Clearly, this class is a quasi-varieties in the language $\mathfrak{L}^*_{gr}$, and free MR-groups, MR-homomorphisms, and so on are defined; moreover, each Abelian MR-group is an R-module and vice versa.

Most of natural examples of exponential group belong to the class $\mathfrak{M}_\mathsf{R}$:

1) An arbitrary group is a $\mathbb{Z}$-group.

2) An Abelian divisible group from $\mathfrak{L}_\mathbb{Q}$ is an $\mathfrak{M}_\mathbb{Q}$-group.

3) A group of period m is a $\mathbb{Z}/m\mathbb{Z}$-group.

4) A module over the ring R is an Abelian MR-group.

5) Free Lyndon R-groups are MR-groups.

6) The exponential nilpotent R-groups over the binomial ring R introduced by P. Hall in [4] are MR-groups.

A systematic study of MR-group was initiated in [5–12]. Results obtained in these papers have turned out to be very useful in solving well-known problems of Tarski.

Below, following [1], we recall some definitions in the category of MR-groups. Let G be an MR-group.

Definition 3. A homomorphism of R-groups $\varphi : G_1 \to G_2$ is called an **R-homomorphism** if $\varphi(g^\alpha) = \varphi(g)^\alpha$, $g \in G$, $\alpha \in \mathsf{R}$.

Definition 4. For $g, h \in G$ and $\alpha \in \mathsf{R}$, the element

$$(g, h)_\alpha = h^{-\alpha} g^{-\alpha} (gh)^\alpha$$

is called the **α-commutator** of the elements g and h.

Clearly, $(gh)^\alpha = g^\alpha h^\alpha (g,h)_\alpha$ and $G \in \mathfrak{M}_\mathsf{R} \iff \big([g,h] = 1 \Longrightarrow (g,h)_\alpha = 1\big)$. This equivalence leads to the definition of an $\mathfrak{M}_\mathsf{R}$-ideal.

Definition 5. A normal R-subgroup $H \trianglelefteq G$ is called an $\mathfrak{M}_\mathsf{R}$-ideal if, for any $g \in G$, $h \in H$ and $\alpha \in \mathsf{R}$,

$$[g,h] \in H \Longrightarrow (g,h)_\alpha \in H.$$

Proposition 1.

(i) *If $\varphi : G_1 \to G_2$ is an R-homomorphism in the category $\mathfrak{M}_R$-groups, then* $\ker\varphi$ *is an $\mathfrak{M}_R$-ideal in G.*

(ii) *If H is an $\mathfrak{M}_R$-ideal in G, then $G/H \in \mathfrak{M}_R$.*

Varieties of an exponential MR-group. Let $X = \{x_i \mid i \in I\}$ be an infinite aplhabet and $F_\mathsf{R}(X)$ let be a free MR-group with free generating set X as an MR-group. Let us call an arbitrary element $w(x_1, \ldots, x_n) \in F_\mathsf{R}(X)$ **R-word** in X. Let G be an MR-group and $g_1, \ldots, g_n \in G$. The map $x_i \mapsto g_i$ can be extended to an R-homomorphism

$\varphi : F_{\mathsf{R}}(X) \to G$. The image of the word $w(x_1,\dots,x_n)^{\varphi} \in G$ under this homomorphism is called value of $w(x_1,\dots,x_n)$ on the elements $g_1,\dots,g_n$. Fix the following notations:

$$w(x_1,\dots,x_n) = w(\overline{x}),\ \ \overline{x} = (x_1,\dots,x_n),\quad w(g_1,\dots,g_n) = w(\overline{g}),\ \ \overline{g} = (g_1,\dots,g_n),$$
$$w(G) = \big\{w(\overline{g}) \mid \ \overline{g} \in G^n\big\} = \big\{w(g_1,\dots,g_n) \mid \ g_i \in G\big\}.$$

Definition 6. An R-word $w(\overline{x})$ is called an **identity on MR-group** G if $w(G) = 1$.

Definition 7. Let W be a subset of $F_{\mathsf{R}}(X)$. Then W define the **variety of MR-groups**

$$\mathfrak{N} = \big\{G \in \mathfrak{M}_{\mathsf{R}} \mid \quad w(G) = e \ \ \forall\, w \in W\big\}.$$

Definition 8. An R-word $u(\overline{x}) \in F_{\mathsf{R}}(X)$ is called a **corollary** of the set of words W, if $u(G) = e$ for any group $G \in \mathfrak{N}$.

Definition 9. The $\mathfrak{M}_{\mathsf{R}}$-ideal of G generated by all values of all words from W is called W-verbal ideal of G.

Let us denote by $W(G)$ the W-verbal ideal of G.

Proposition 2. *A verbal ideal in $F_{R}(X)$ generated by the set of word W consists exactly of all corollaries of the set W in $F_{R}(X)$.*

Definition 10. A group $F_{W,\mathsf{R}}(X) \in \mathfrak{N}$ is called a *free group with the base X in the varieties* $\mathfrak{N}$ if $F_{W,\mathsf{R}}(X)$ R-generated by the set X and for any group $G \in \mathfrak{N}$ arbitrary map $\varphi_0 : X \to G$ can be extended to an R-homomorphism $\varphi : F_{W,\mathsf{R}}(X) \to G$.

Theorem 1. *The group $F_{R}(X)/W(F_{R}(X))$ is a free group in the varieties of $\mathfrak{N}$ which is defined by W.*

Theorem 2. *A class of MR-groups $\mathfrak{N}$ is varieties if $\mathfrak{N}$ is closed with respect to taking subgroups, Cartesian products and R-homomorphisms.*

The proof of the last theorem is the same as the Birkhoff's one for varieties of algebraic systems.

Definition 11. The subgroup $(G,G)_{\mathsf{R}} = \langle (g,h)_{\alpha} \mid \quad g,h \in G,\ \ \alpha \in \mathsf{R}\rangle_{\mathsf{R}}$ of G is called the MR-commutant of G.

Theorem 3. *For any MR-group G the following is true:*

(i) *The R-commutant of G is a R-subgroup of G by all commutator $[x,y] = x^{-1}y^{-1}xy$.*

(ii) *The R-commutant is the smallest $\mathfrak{M}_{R}$-ideal of G among all ideal H such that G/H is an Abelian MR-group.*

Theorem 4. *Let R be a field. Then the word $(x,y)_{\alpha} = y^{-\alpha}x^{-\alpha}(xy)^{\alpha}$ generates R-commutant as a verbal subgroup if $\alpha \neq 0,-1$.*

Theorem 5. *Any R-words subset V of $F_{R}(X)$ is equivalent to a set of R-words*

$$W = \Big\{x_1^{\alpha_i},\ u_j \mid \ i \in I,\ j \in J,\ \alpha_i \in R,\ u_j \in \big(F_{R}(X), F_{R}(X)\big)_{R}\Big\}$$

for suitable sets of indexes I and J.

Remark. When defining the varieties of $\mathfrak{M}_{\mathsf{R}}$-groups we follow V. A. Gorbunov's monograph [13], which declares how one can understand varieties of groups inside, quasi-varieties of groups. Therein it is shown that for these varieties all the well-known Birkhoff theorems are that for them there exists the notion of a free group and the theory of defining relations.

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