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TO THERMOELASTIC WAVES

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Abstract. In this work we construct some expressions having resolving significance for creation of new two-dimensional mathematical models of *von Kármán-Mindlin-Reissner* type systems of partial differential equations for thermo-dynamic elastic plates of finite thickness of heat conducting isotropic material. Our models contain some members described particularly physical motions named as thermoelastic and solitons type waves.

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In this report we give some expressions having resolving significance for constructing new two-dimensional mathematical models of *von Kármán-Mindlin-Reissner* type systems of partial differential equations (PDEs) for thermo-dynamic elastic plates of finite thickness of heat conducting isotropic material. Our models contain some members described particularly new physical motions which may be named as thermoelastic and solitons type waves. As the basic equations we choose the dynamic system of nonlinear theory of elasticity [1,2] with heat conductions [3-7].

Thus the PDEs governing nonlinear thermo-dynamical elastic interactions are:

$$\partial_j\left(\sigma_{ij}+\sigma_{kj}u_{i,k}\right)=f_i+\rho\,\partial_{tt}u,\ (x,t)\in Q_T=\Omega_h\times(0,T)\,,\ \Omega_h=D(x,y)\times\,]-h,h[\,,\quad(1)$$

$$\varepsilon_{ij}=0.5(u_{i,j}+u_{j,i}+u_{k,i}u_{k,j}),\quad \sigma_{ij}=(\lambda\varepsilon_{kk}-\beta_i\vartheta)\delta_{ij}+2\mu\varepsilon_{ij},\quad(2)$$

$$K_i\left(\partial_i\left(1+\epsilon(\vartheta)\right)\partial_i\vartheta\right)=\left(\partial_t+\tau_0\partial_{tt}\right)\left(a\vartheta+b_i\varepsilon_{ii}\right)+f_4,\quad(3)$$

$$\sigma_{i3}+\sigma_{j3}u_{i,j}=g_i^{\pm},\ x_3\in S^{\pm}=D\times\{\pm h\}\,;l[u]=g,\ x\in S=\partial D\times\,]-h,h[\,,\quad(4)$$

$$(\alpha^{\pm}\pm\partial_3)\vartheta(x,y,\pm h,t)=g_4^{\pm},\ l_1\vartheta=g_4,\ x\in S=\partial D\times\,]-h,h[\,,\quad(5)$$

$$u\,(x,0)=\varphi_0,\ \partial_t u(x,0)=\varphi_0',\ \vartheta(x,0)=\phi_0,\partial_t\vartheta(x,0)=\phi_0'.\quad(6)$$

Here u, ε, σ, ϑ are unknown values, $f=(f_1,f_2,f_3)^T$, f_4, g, $g^{\pm}$, φ_0, φ_0', ϕ_0, ϕ_0', ρ are given vectors and functions respectively, the numbers a, b_i, $\alpha^{\pm}\geq 0$, β_i, τ_0, K_i are defined (see [2] pp.6-16, [6, 7]).

Below using ([2], pp.6-19) we investigate the problem of creation of mathematical models corresponding to refined theories of *von Kármán* type for (1)-(6) initial-boundary value problem when $\epsilon=\epsilon(\vartheta)$ is a linear function. ϵ is a small parameter. We assume that the investigation of same problems represent some interest and in this connection we visible the works where analogous problems are considered. In the well-known monographs of Kupradze with co-authors and Novatskii, Chandrasekharaian [6], Verma [7] linear problems are considered when for thin-walled thermoelastic plates D represents a plane, a half-plane or a quadrant. Using Fourier, Laplace transformations respectively, the original problems are reduced to two-dimensional initial-boundary value problems. For finite plates linear problems of thermoelasticity were investigated by Kovalenko. He published monograph where he presupposed ([3], p.74): "that for problems corresponding to Extension-Compression (Ex-Com) phenomenon is true that a stationary temperature field without thermal sources doesn't provoke horizontal components of stress tensor". In [2, pages 121-124] we prove that this result contains the contradiction. We constructed two-dimension Helmholtz DE, the presence of which proves that in case of an elastic plate even without thermal sources in Ω_h there may arise important distributions of a temperature, that denotes quantitive changes of the right-side of an equilibrium equation (1).

Thus the averaged equations using ([2], p.36) give us the following forms:

$$\begin{gathered}\left(D\Delta^2 + 2h\rho\partial_{tt} - 2DE^{-1}(1+\nu)\rho\partial_{tt}\Delta\right)w = \left(1 - \frac{h^2(1+2\gamma)(2-\nu)}{3(1-\nu)}\Delta\right) \\ \times\left(g_3^+ - g_3^-\right) + 2h\left(1 - \frac{2h^2(1+2\gamma)}{3(1-\nu)}\Delta\right)\left([w,\varphi] - \beta_{3-\alpha}\bar{\vartheta}\partial_{\alpha\alpha}w\right) \\ + \frac{2h^3}{3}\left(\beta_\alpha\partial_{\alpha\alpha} - 2\beta_3\right)\vartheta^* - \beta_3\left(g_4^+ - g_4^-\right) + h\left(g_{\alpha,\alpha}^+ + g_{\alpha,\alpha}^-\right) \\ - \int_{-h}^{+h}\left(zf_{\alpha,\alpha} - \left(1 - \frac{1}{1-\nu}\Delta\left(h^2 - z^2\right)\right)f_3\right)dz,\end{gathered} \tag{7}$$

$$\begin{gathered}\left(\Delta^2 - \frac{1-\nu^2}{E}\rho\Delta\partial_{tt}\right)\varphi = -\frac{E}{2}[w,w] + 2\beta_\alpha\partial_{\alpha\alpha}\bar{\vartheta} \\ -\partial_\alpha\left([\bar{u}_\alpha,\varphi] - \bar{u}_{\alpha,\alpha}\left(\partial_\alpha(\partial_{\alpha\alpha} - \partial_{3-\alpha 3-\alpha})\varphi\right) + \beta_\gamma\partial_\gamma(\bar{\vartheta}\bar{u}_{\alpha,\gamma}\right) \\ + \frac{\nu}{2}\left(\Delta - \frac{2\rho}{E}\partial_{tt}\right)\left(g_3^+ + g_3^-\right) + \frac{1+\nu}{2h}(f_{\alpha,\alpha} - g_{\alpha,\alpha}^+ + g_{\alpha,\alpha}^-).\end{gathered} \tag{8}$$

Rewrite the equation (3) in the convenient form:

$$M[\vartheta] = K_i\partial_i\left((1+\epsilon\theta)\partial_i\vartheta\right) = (\partial_t + \tau_0\partial_{tt})(a\vartheta + b\varepsilon) + f_4 = F\left[a\vartheta + bu\right] + f_4. \tag{9}$$

If we apply now the methodology ([2], pp.103-110, 134), follows:

$$J_1\left[M[\vartheta]\right] = \int_{-h}^{h} K_\varepsilon\partial_\varepsilon\left((1+\epsilon\theta)\partial_\alpha\vartheta\right)dz = hK_\alpha\partial_{\alpha\alpha}(2\bar{\vartheta} + \epsilon(\vartheta_0^2 + \frac{1}{3}\vartheta_1^2 + \frac{1}{5}\vartheta_2^5 + ...)), \tag{10}$$

$$\begin{gathered}J_2\left[M[\vartheta]\right] = \int_{-h}^{h} K_\alpha\partial_\alpha\left((1+\epsilon\theta)\partial_\alpha\vartheta\right)P_1(z/h)dz \\ = K_\alpha\left(\int_{-h}^{h}\frac{z}{h}\partial_{\alpha\alpha}\vartheta dz + \epsilon\int_{-h}^{h}\frac{z}{2h}\partial_{\alpha\alpha}(\vartheta^2)dz\right) \\ = \frac{2h}{3}K_\alpha\partial_{\alpha\alpha}\left(\vartheta_1 + \epsilon\sum_{k=0}^{\infty}\frac{1}{2k+1}\vartheta_k\left(\frac{k}{2k-1}\vartheta_{k-1} + \frac{k+1}{2k+3}\vartheta_{k+1}\right)\right),\end{gathered} \tag{11}$$

$$J_k[M[\vartheta]] = \int\limits_{-h}^{h} K_\alpha \partial_\alpha \left((1+\epsilon\theta)\partial_\alpha \vartheta\right) P_k(z/h) dz$$
$$= hK_\alpha \partial_{\alpha\alpha} \left(\frac{2}{2k+1}\vartheta_k + \frac{\epsilon}{2} \int\limits_{-1}^{1} \sum_{m=0}^{\infty} \vartheta_m \sum_{n=0}^{\infty} \vartheta_n \sum_{r=0}^{\min(n,k)} \alpha_{nkr} P_{n+k-2r}(u)\, du \right). \qquad (12)$$

where

$$\alpha_{nkr} = \frac{A_{n-r}A_r A_{k-r}}{A_{n+k-r}} \frac{2(n+k)-4r+1}{2(n+k)-2r+1}, \quad A_r = \frac{(2r-1)!}{r!}$$

It is evident that we must calculate also the scalar products with $K_3\partial_3\left((1+\epsilon\vartheta)\,\partial_3\vartheta\right)$ as well as with respect to dynamical parts of (3). Let us consider boundary conditions (5_1). We have two different cases: $\alpha^\pm = 0$; $\alpha^+ + \alpha^- > 0$.

For the first case we have:

$$J_k^\pm = \int\limits_{-h}^{h} P_k\left(\frac{z}{h}\right) K_3\partial_3\left((1+\epsilon\vartheta)\,\partial_3\vartheta\right) dz, \quad k = 0, 1, ...$$

$$J_0^\pm = K_3\left[g_4^+\left(1+\epsilon\vartheta\,(x,y,h,t)\right) - g_4^-\left(1+\epsilon\vartheta\,(x,y,-h,t)\right)\right], \qquad (13)$$

$$J_1^\pm = 2hK_3\left[g_4^+\left(1-\vartheta\,(x,y,h,t)\right) + g_4^-\left(1-\vartheta\,(x,y,-h,t)\right)\right]$$
$$+\frac{\epsilon}{2}K_3\int\limits_{-h}^{h}\frac{z}{h}\partial_{33}\left(\vartheta^2\right)dz = 2hK_3\left[g_4^+\left(1-\vartheta\,(x,y,h,t)\right) + g_4^-\left(1-\vartheta\,(x,y,-h,t)\right)\right]$$
$$+\epsilon K_3\left(\left(\vartheta\partial_3\,\vartheta - h^{-1}\vartheta^2\right)\big|_{\pm h}\right) \qquad (14)$$
$$\approx 2K_3\Big(h\left[g_4^+\left(1-\vartheta\,(x,y,h,t)\right) + g_4^-\left(1-\vartheta\,(x,y,-h,t)\right)\right]$$
$$+\left[\left(\vartheta_0+\vartheta_1\right)g_4^+ + \left(\vartheta_0-\vartheta_1\right)g_4^-\right] - \tfrac{2}{h}\vartheta_0\vartheta_1\Big).$$

We must remark that $\vartheta_0 = \bar{\vartheta}, \vartheta_1 = \vartheta^*$.

For the second case we use ([2],pages 121-124). Instead of formula (10.2) we have:

$$\vartheta(x,y,z,t) = \frac{1}{2+\alpha_1+\alpha_2}\Big[\left(1+\alpha_1\right)\beta_2 + \left(1+\alpha_2\right)\beta_1 + z(\beta_2-\beta_1)$$
$$+\int\limits_{-1}^{z}\left(\left(1+\alpha_2-u\right)\left(1+\alpha_1+z\right) + \left(z-u\right)\left(2+\alpha_1+\alpha_2\right)\right)\partial_{33}\vartheta du \qquad (15)$$
$$+\int\limits_{z}^{1}(z-u)(2+\alpha_1+\alpha_2)\partial_{33}\vartheta du\Big].$$

For our case we have $\alpha_1 = \alpha^-$, $\alpha_2 = \alpha^+$, $\beta_1 = g_4^-$, $\beta_2 = g_4^+, h = 1$.

The last expressions are devoted to discovering new nonlinear members which are neglected usually even in the prime case when the medium is an isotropic elastic plate with constant thickness.These members are same to Monge-Amper operator and correspond to Ex-Com problems. In [8] instead of second differential equation relatively of Airy function of *von Kármán* system we constructed the PDEs (2.7-9,11), corresponding to processes of Ex-Com. Here the right parts $\bar{f}_\alpha$ contain the summands depending on $\partial_\beta(\sigma_{k\beta}u_{\alpha,k})$ products too. If we now used the first two equations of (1) the corresponding system of averaged PDEs for Ex-Com would also have the members of following form (compere with (8)):

$$S_\alpha = \int_{-h}^{h} \partial_\beta(\sigma_{k\beta}u_{\alpha,k})dz \approx 2h([\bar{u}_\alpha, \varphi] + \bar{u}_{\alpha,\alpha}\,(\partial_{\alpha\alpha} - \partial_{3-\alpha,3-\alpha})\,\partial_\alpha\varphi.$$

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