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ON THE CONVERGENCE OF FOURIER INTEGRAL MEANS OF FUNCTIONS DEFINED ON LOCALLY COMPACT ABELIAN GROUPS

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Abstract. The problem of the pointwise convergence of special Fourier integral means of integrable with respect to a Haar measure functions defined on a locally compact the Abelian group is studied. Some examples illustrating the results are considered.

Keywords and phrases: Locally compact Abelian groups, dual groups, Fourier transform, Fourier integral means, Fejér integral, characteristic function.

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Let G be a locally compact Abelian Hausdorf group G and let $\widehat{G}$ be its dual group, i.e. the set of all characters on G. $\widehat{G}$ is also a locally compact Abelian group in the topology of uniform convergence of characters on compact subsets of G. $U_{\widehat{G}}$ will stand for the collection of all symmetric compact sets from $\widehat{G}$ which are closures of neighborhoods of the unity in $\widehat{G}$. In papers [1-6] problems of approximate nature are considered for some spaces of real or complex valued functions and also measures defined on G.

Let I be an ordered unbounded set in $\mathbb{R}^+$ and consider a generalized sequence of sets $K_\alpha \in U_{\widehat{G}}$, such that $K_\alpha \subset K_\beta$ if $\alpha < \beta$ $(\alpha, \beta \in I)$ and $\cup_{\alpha \in I} K_\alpha = \widehat{G}$. For functions of $L^p(G)$, $1 \le p \le \infty$, i.e. for p-th power integrable functions on G with respect to the Haar measure μ ($L^\infty(G)$ denotes the space of essentially bounded on G functions with respect to μ) we have studied in [5-6] the following sequence of positive operators

$$\sigma_{K_\alpha}(f)(g) \equiv (f * V_{K_\alpha})(g) = \int_G f(h) V_{K_\alpha}(h^{-1}g)dh, \tag{1}$$

where

$$V_{K_\alpha}(g) = (\mathrm{mes} K_\alpha)^{-1} (\widehat{(1)_{K_\alpha}}(g))^2, \quad K_\alpha \in U_{\widehat{G}}. \tag{2}$$

and $\widehat{(1)_{K_\alpha}}$ is the Fourier transform of the characteristic function $(1)_{K_\alpha}$ of the set K_α. We suppose that the Haar measure on G and $\widehat{G}$ are normalized so that the inversion formula hold for functions $f \in L^1(G)$, $\widehat{f} \in L^1(\widehat{G})$.

With the help of well-known properties of the Fourier transform (see, for example, [7]) we obtain for the functions $f \in L^1(G)$

$$\sigma_{K_\alpha}(f)(g) = (\mathrm{mes} K_\alpha)^{-1} \int_G f(h) \widehat{(1)_{K_\alpha}}(h^{-1}g) \widehat{(1)_{K_\alpha}}(h^{-1}g)dh$$

$$= (\mathrm{mes} K_\alpha)^{-1} \int_G f(h) (1)\widehat{_{K_\alpha} * (1)}_{K_\alpha}(h^{-1}g)dh$$

$$= (\mathrm{mes} K_\alpha)^{-1} \int_{\widehat{G}} \widehat{f}(\chi)((1)_{K_\alpha} * (1)_{K_\alpha})(\chi^{-1})\chi(g)d\chi.$$

This means that if $f \in L^1(G)$, then defined by (1) operator $\sigma_{K_\alpha}(f)(g)$ can be written in the form of Fourier integral means.

It is proved in [3] that if $f \in L^p(G),\ 1 \le p < \infty$, and the sequence of sets $K_\alpha \in U_{\widehat{G}}$ satisfies the condition

$$\lim_{\alpha\to\infty} \frac{\text{mes}(TK_\alpha)}{\text{mes}K_\alpha} = 1 \tag{3}$$

for all fixed nonempty $T \in U_{\widehat{G}}$, then

$$\lim_{\alpha\to 0} \|f - \sigma_{K_\alpha}(f)\|_{L^p(G)} = 0. \tag{4}$$

Here we study the problem of pointwise convergence of the operator σ_{K_α} for $f \in L^1(G)$ in the points of continuity of f. For the functions belonging to L^∞, in [3] it is proved that if f is continuous at a neighborhood of a set $S \subset G$ and the condition (3) is satisfied, then $\sigma_{K_\alpha}(f)$ converges uniformly on S as $\alpha \to \infty$. This result follows from the following approximate property of the kernels V_{K_α} [5]: If U is a nonempty element of $U_{\widehat{G}}$, then

$$\lim_{\alpha\to\infty} \int_{G\setminus U} V_{K_\alpha}(g)dg = 0. \tag{5}$$

It is interesting to remark in connection with this relation that convergence to the zero in (5) holds even if mesK_α does not converges to the infinity if $\alpha \to \infty$. For example, if G is the group Z of whole numbers with respect to the addition, then $\widehat{G}$ is the unit circle, which is a compact Abelian group with the operation of multiplication of complex numbers. If

$$K_\alpha = \{e^{i\theta} :\ -\pi + \pi/\alpha \le \theta \le \pi - \pi/\alpha\}$$

and $n \in Z$, then

$$\widehat{(1)_{K_\alpha}}(g) = (i\sqrt{2\pi})^{-1} \int_{K_\alpha} \xi^n d\xi$$

$$= \sqrt{2/\pi}n^{-1}\sin(\pi - \pi/\alpha)n, \quad \text{mes}K_\alpha = 2\pi(\alpha - 1)/\alpha$$

and for any fixed $m \ge 1$

$$\lim_{\alpha\to\infty} \int_{|n|\ge m} V_{K_\alpha}(n)dn = \frac{2\alpha}{\pi^2(\alpha-1)} \sum_{n=m}^{\infty} \lim_{\alpha\to\infty} (\sin^2(\pi - \pi/\alpha)n)/n^2 = 0.$$

Theorem. *Let K_α be a sequence in $U_{\widehat{G}}$ satisfying (3), S is a compact set of G and U be a nonempty set from $U_{\widehat{G}}$. If a function $f \in L^1(G)$ is uniformly continuous at a neighborhood of S and the kernels V_{K_α} are uniformly bounded relative to $\alpha \in I$ on $G\setminus U$, i.e.*

$$\lim_{\alpha} \sup_{g\in G\setminus U} V_\alpha(g) \le C, \tag{6}$$

where C depends only on U, then $\sigma_{K_\alpha}(f)(g)$ converges to $f(g)$ uniformly on S.

Proof. Since f is uniform continuous on $S \in G$, given $\varepsilon > 0$ find a neighborhood $U \in U_{\widehat{G}}$ such that

$$|f(hg) - f(g)| < \varepsilon \quad \text{for all } g \in S, h \in U. \tag{7}$$

According to the equality $\int_G V_{K_\alpha}(g)dg = 1$ [6], we have

$$f(g) - \sigma_{K_\alpha}(f)(g) = \int_G (f(g) - f(h^{-1}g))V_{K_\alpha}(h)dh$$

$$= \int_U + \int_{G\setminus U} = I_1 + I_2. \tag{8}$$

It follows from (7) that

$$I_1 \leq \varepsilon \int_G V_{K_\alpha}(g)dg = \varepsilon. \tag{9}$$

According to (4) there exists a set $T \in U_{\widehat{G}}$, for which

$$\|f - f * V_T\|_{L^1(G)} < \varepsilon \tag{10}$$

If $\varphi \equiv f * V_T$, then

$$|\varphi(g)| = |\int_G f(h^{-1}g)V_T(h)dh| \leq \|f\|_{L^1(G)}\|V_T\|_{L^\infty}(G).$$

The Fourier transform of $V_T \in L^1(G)$ has compact support in $\widehat{G}$. In [1] it is proved that functions of such type belong to the space L^∞ too. Thus $\varphi = f * V_T \in L^1(G) \cap L^\infty(G)$. Applying this fact we obtain

$$I_2 \leq \int_{G\setminus U} |f(h^{-1}g) - \varphi(h^{-1}g)|V_{K_\alpha}(h)dh$$

$$+ \int_{G\setminus U} |\varphi(h^{-1}g)|V_{K_\alpha}(h)dh + |f(g)| \int_{G\setminus U} V_{K_\alpha}(h)dh$$

$$\leq \lim_\alpha \sup_{g\in G\setminus U} V_{K_\alpha}(g)\|f - \varphi\|_{L^1(G)} + (\sup_{g\in G} |\varphi(g)| + |f(g)|) \int_{G\setminus U} V_{K_\alpha}(h)dh.$$

From here and relations (5)-(10), follows the validity of theorem. The condition (6) is valid for all groups, considered by us in [1]-[3]. For example, let $G = Q_p$ be the field of the p-adic numbers for a prime p. It is well known that Q_p is a locally compact Abelian group with respect to the addition operation and its dual group $\widehat{Q_p}$ is isomorphic to this addition group ([8], p. 50). For any fixed $\xi \in Q_p$ the additive character of additive group Q_p has the form $\chi_\xi(x) = \exp\{2\pi i\{\xi x\}_p\}$, where $\{\xi x\}_p$ is a fractional part of $\xi x \in Q_p$ ([8], p. 48). Let $n \in \mathbb{N}$ and K_n be the p-adic ball of the radius p^n with the center in zero, i.e. $K_n = \{h \in Q_p : \ |h|_p \leq p^n\}$, where $|h|_p$ is the p-adic norm of h. The Haar measure of this ball is $\mathrm{mesK_n} = p^n$ ([8], p. 59). In [8] (p. 62) it is proved that $\widehat{(1)_{K_n}}(\xi) = p^n$, if $|\xi|_p \leq p^{-n}$, and $\widehat{(1)_{K_n}}(\xi) = 0$ if $|\xi|_p > p^{-n+1}$. It follows from here that if $\xi \neq 0$ and $n \to \infty$, then condition (6) is fulfilled. It is clear that this condition is true for the above mentioned group $G = Z$.

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