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SOLUTION OF SOME BOUNDARY VALUE PROBLEM OF STATICS OF THE THEORY OF ELASTIC MIXTURE FOR A CIRCLE

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Abstract. For the two-dimensional homogeneous equation of statics of the linear theory of elasitc mixture, in the case a circle we consider the two boundary value problem which is analogous to III and IV interior baundary value problem of the classic theory of elastisity. On the basis of formulas analogous to Kolosov-Muskhelishvili our problems are reduced to the Riemann-Hilbert problems for a circle, and owing to the above result, the solution of the problems can be reduced to the first order linear differential equations.

Keywords and phrases: Elastic mixture, Riemann-Hilbert problem. Kolosov-Muskhelishvili type formulas.

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1^0 The homogeneous equation of statics of the linear theory of elastic mixture in the complex form is written as [2]

$$\frac{\partial^2 U}{\partial z \partial \overline{z}} + K \frac{\partial^2 \overline{U}}{\partial \overline{z}^2} = 0, \qquad U = \begin{pmatrix} u_1 + iu_2 \\ u_3 + iu_4 \end{pmatrix}, \tag{1}$$

where $u_p, p = \overline{1,4}$ are components of the displacement vector, $z = x_1 + ix_2$,

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right), \qquad K = -\frac{1}{2}em^{-1},$$

$$e = \begin{bmatrix} e_4 & e_5 \\ e_5 & e_6 \end{bmatrix}, \qquad m^{-1} = \frac{1}{\triangle_0}\begin{bmatrix} m_3 & -m_2 \\ -m_2 & m_1 \end{bmatrix}, \qquad \triangle_0 = m_1 m_3 - m_2^2,$$

$m_k, e_{3+k}, k = 1, 2, 3$ are expressed in terms of elastic constants [2].

In [2] M. Basheleishvili obtained the representations (Kolosov-Muskhelishvili type formulas)

$$U = (u_1 + iu_2, u_3 + iu_4)^T = m\varphi(z) + \frac{1}{2}ez\overline{\varphi'(z)} + \overline{\psi(z)}, \tag{2}$$

$$TU = \begin{pmatrix} (TU)_2 - i(TU)_1 \\ (TU)_4 - i(TU)_3 \end{pmatrix} = \frac{\partial}{\partial S(x)}\left[(A - 2E)\varphi(z) + Bz\overline{\varphi'(z)} + 2\mu\overline{\psi(z)}\right], \tag{3}$$

where $\varphi = (\varphi_1, \varphi_2)^T$ and $\psi = (\psi_1, \psi_2)^T$ are arbitrary analytic vector-functions, $(TU)_p, p = \overline{1,4}$, are components of the stress [1], $\frac{\partial}{\partial S(x)} = n_1\frac{\partial}{\partial x_2} - n_2\frac{\partial}{\partial x_1}, n = (n_1, n_2)^T$ is the unit vector of the outer normal, and E is the unit matrix;

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} = 2\mu m, \quad B = \mu e, \quad m = \begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix}, \quad \mu = \begin{bmatrix} \mu_1 & \mu_3 \\ \mu_3 & \mu_2 \end{bmatrix}.$$

Here (see[1])

$$\Delta_0 = detm > 0, \Delta_1 = det\mu > 0, \Delta_2 = det(A-2E) > 0,$$
$$0 < A_1 + A_4 < 4, A_1 + A_4 - 4\Delta_0\Delta_1 > 0, (A_1 + A_4)^2 - 16\Delta_0\Delta_1 > 0. \quad (4)$$

Introduce the vectors:

$$U_n = \begin{pmatrix} u_1n_1 + u_2n_2 \\ u_3n_1 + u_4n_2 \end{pmatrix}, U_s = \begin{pmatrix} u_2n_1 - u_1n_2 \\ u_4n_1 - u_3n_2 \end{pmatrix}, \sigma_n = \begin{pmatrix} (TU)_1n_1 + (TU)_2n_2 \\ (TU)_3n_1 + (TU)_4n_2 \end{pmatrix},$$

$$\sigma_s = \begin{pmatrix} (TU)_2n_1 - (TU)_1n_2 \\ (TU)_4n_1 - (TU)_3n_2 \end{pmatrix}.$$

2^0 Let $D^+ = \{z : |z| < 1\}$ and $L = \{z : |z| = 1\}$. We consider the problems. Find, in the domain D^+, a vector $U = (u_1+iu_2, u_3+iu_4)^T$ which belongs to the class $C^2(D^+)\cap C^{1,\alpha}(D^+\cup L)$, is a solution of equation (1) and satisfying one of the following boundary conditions on L

$$2\mu(U_n(t))^+ = f^{(1)}(t), (\sigma_s(t))^+ = F^{(1)}(t), Problem(III^*)^+,$$
$$2\mu(U_s(t))^+ = f^{(2)}(t), (\sigma_s(t))^+ = F^{(2)}(t), Problem(IV^*)^+,$$

where $f^{(j)}$ and $F^{(j)}(j = 1,2)$ are real given vector-functions on L, satisfying certain conditions.

Using the Green formula [1] it is easy to prove.

Theorem 1. Solution of the BVP $(III^*)^+$ is not unique. In this case two regular solutions of the problem $(III^*)^+$ differ by a rigid rotation, $(U = (i\varepsilon z, i\varepsilon z)^T$, where ε is an arbitrary constant).

Theorem 2. The boundary value problem $(IV^*)^+$ has a unique regular solution.

3^0 Consider the BVP $(III^*)^+$. Using formulas (2) and (3) the boundary conditions can be written as follows:

$$Re\left\{e^{-i\theta}[A\varphi(t) + Bt\overline{\varphi'(t)} + 2\mu\overline{\psi(t)}\right\}^+ =$$
$$= Re\left[At^{-1}\varphi(t) + B\varphi^{'}(t) + 2\mu t\psi(t)\right]^+ = f^{(1)}(t), t \in L, \quad (5)$$
$$Re\left\{e^{-i\theta}\frac{\partial}{\partial s(t)}[(A-2E)\varphi(t) + Bt\overline{\varphi'(t)} + 2\mu\overline{\psi(t)}]\right\}^+ =$$
$$Re\left\{i[(A-B-2E)\varphi^{'}(t) + Bt\varphi^{''}(t) + 2\mu t^2\psi^{'}(t)]\right\}^+ = F^{(1)}(t), t \in L, \quad (6)$$

where $t = e^{i\theta}$ is the polar equation of L. $f^{(1)} \in C^{1,\alpha}(L)$, $F^{(1)} \in C^{0,\alpha}(L)$, $0 < \alpha < 1$, $\int\limits_L F^{(1)}dS = 0$.

Now note that, since $\varphi(z)$ and $\psi(z)$ are arbitrary analytic vector-functions, we can suppose that

$$\varphi_j(z) = \sum_{n=2}^{\infty} A_n^{(j)} z^n, \psi_j(z) = \sum_{n=0}^{\infty} B_n^{(j)} z^n, |z| < 1, j = 1,2, \quad (7)$$

where $A_n^{(j)}$ and $B_n^{(j)}$ are constant $(j = 1,2)$.

Owing to the above reasoning, we can conclude that the boundary conditions (5) and (6) are the Riemann - Hilbert problems for a circle $|z| < 1$.

A solution of problems (5) and (6) can be represented in the form respectively (see [3])

$$Az^{-1}\varphi(z)+B\varphi^{'}(z)+2\mu z\psi(z)=\frac{1}{2\pi i}\int\limits_{L}\frac{(t+z)f^{(1)}(t)}{t(t-z)}dt+iC_1, \tag{8}$$

$$(A-B-2E)\varphi^{'}(z)+Bz\varphi^{''}(z)+2\mu z^2\psi^{'}(z)=-\frac{1}{2\pi}\int\limits_{L}\frac{(t+z)F^{(1)}(t)}{t(t-z)}dt+C_2, \tag{9}$$

where C_1 and C_2 are arbitrary real constant vectors. Rearing in mind (7) we arrive at the conclusion

$$C_1=\frac{1}{2\pi}\int\limits_{L}\frac{f^{(1)}(t)dt}{t}, C_2=\frac{1}{2\pi}\int\limits_{L}\frac{F^{(1)}(t)dt}{t}=0. \tag{10}$$

Substituting (10) in (8) and (9) we obtain

$$A\varphi(z)+Bz\varphi^{'}(z)+2\mu z^2\psi(z)=\frac{z^2}{\pi i}\int\limits_{L}\frac{f^{(1)}(t)dt}{t(t-z)}, \tag{11}$$

$$(A-B-2E)\varphi^{'}(z)+Bz\varphi^{''}(z)+2\mu z^2\psi^{'}(z)=-\frac{z}{\pi}\int\limits_{L}\frac{f^{(1)}(t)dt}{t(t-z)}, \tag{12}$$

The relations (11) and (12) yield

$$(B+E)\varphi^{'}(z)+2\mu z\psi(z)=\frac{z^2}{2\pi i}\int\limits_{L}\frac{f^{(1)}(t)dt}{t(t-z)^2}+\frac{z}{2\pi i}\int\limits_{L}\frac{2f^{(1)}(t)+iF^{(1)}(t)}{t(t-z)}dt. \tag{13}$$

From(11) and (13) we have

$$\varphi^{'}(z)-A\frac{1}{z}\varphi(z)=g(z), g(z)=\frac{z^2}{2\pi i}\int\limits_{L}\frac{f^{(1)}(t)dt}{t(t-z)^2}+\frac{z}{2\pi}\int\limits_{L}\frac{F^{(1)}(t)dt}{t(t-z)}. \tag{14}$$

Combining equalities (11) and (14), we find that

$$\psi(z)=\frac{\mu^{-1}}{2}\times$$

$$\times\left[\frac{1}{\pi i}\int\limits_{L}\frac{f^{(1)}(t)dt}{t(t-z)}-\frac{B}{2\pi}\int\limits_{L}\frac{F^{(1)}(t)dt}{t(t-z)}-\frac{Bz}{2\pi i}\int\limits_{L}\frac{f^{(1)}(t)dt}{t(t-z)^2}-(A+BA)\varphi(z)\right]. \tag{15}$$

It follows from (15) that the solution of the Problem $(III^*)^+$ is reduced to finding solution of the differential equation (14).

We write equation (14) in the form

$$\begin{pmatrix}1\\ y\end{pmatrix}\varphi^{'}(z)-\frac{A_1+A_3y}{z}\begin{pmatrix}1\\ y\end{pmatrix}\varphi(z)=\begin{pmatrix}1\\ y\end{pmatrix}g(z), \tag{16}$$

where y is an arbitrary real constant. Define y by the equation

$$y(A_3y+A_1)=A_4y+A_2. \tag{17}$$

Finally, the solution of equation (14) can be represented as follows

$$\varphi(z)=\frac{1}{y_2-y_1}\begin{bmatrix} y_2 z^{\gamma_1} & -y_1 z^{\gamma_2} \\ -z^{\gamma_1} & z^{\gamma_2}\end{bmatrix}\int\begin{bmatrix} z^{-\gamma_1} & 0 \\ 0 & z^{-\gamma_2}\end{bmatrix}\begin{bmatrix} 1 & y_1 \\ 1 & y_2\end{bmatrix} g(z)dz \qquad (18)$$

where y_1 and y_2 are roots of the equation (17)

$$\gamma_j=\frac{1}{2}[A_1+A_4-(-1)^j\sqrt{(A_1+A_4)^2-16\Delta_0\Delta_1}>0, j=1,2,(see(4)).$$

Substituting in formula (2) the value $\psi(z)$ appearing in (15) and the value $\varphi(z)$ appearing in (18) we obtain the solution (in quadrature) of Problem $(III^*)^+$.

4^0 The BVP $(IV^*)^+$ is solved quite analogously.

R E F E R E N C E S

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