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TO NUMERICAL REALIZATIONS AND STABILITY OF CALCULATING PROCESS OF SOME PROBLEMS OF THEORY OF ELASTICITY FOR CROSS-SHAPED REGIONS

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Abstract. New algorithms of the approached decision of Poisson equation (Dirichlet boundary problem) for a two-dimensional crosswise body by means of Schwartz iterative method are considered. The unknown function expands into the Fourier-Legendre series. Differences of Legendre polynomial are used as basic functions. The five-dot linear system of the algebraic equations concerning unknown coefficients is received. The program code (on the basis of Matlab) for the approached decision of the considered problem is created; corresponding numerical experiments are made which revealed stability of the account process.

Keywords and phrases: Poisson equation, Dirichlet boundary value problem, Galerkin method, Schwartz method, Fourier - Legendre series.

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1. Introduction. Methods of the decision of the mathematical physics equations, in particular, elasticity theories, both exact, and approximate, are given in different scientific articles and monographs. The approximate methods are developed for search of the intense-deformed elastic bodies having difficult geometry of different types (see for example [1]-[3]). Now it is a very actual problem working out and search of the designs having difficult geometry, definition of the intense-deformed state of a body, creation of corresponding settlement algorithms and software products.

2. Statement of the problem. Let's solve a Dirichlet problem for the Poissons equation:

$$\Delta u(x,y) = f(x,y), \quad (x,y) \in \Omega, \tag{1}$$

$$u(x,y) = g(x,y), \quad (x,y) \in \partial\Omega, \tag{2}$$

where $u(x,y) \in C^2(\Omega)$ is unknown, $f(x,y) \in C(\Omega)$, $g(x,y) \in C(\partial\Omega)$ are given functions,

$\Omega = \Omega_1 \cup \Omega_2$ is a given body,
$\Omega_1 = \{(x,y) : -2 \le x \le 2,\ -1 \le y \le 1\}$; $\Omega_2 = \{(x,y) : -2 \le y \le 2,\ -1 \le x \le 1\}$,
$\partial\Omega = \Gamma = \Gamma_1 \cup \Gamma_2$ is a boundary of the given body
$\Gamma_1 = \{(x,y) : y = \pm 1,\ -2 \le x \le -1 \text{ or } 1 \le x \le 2;\ x = \pm 2,\ -1 \le y \le 1\}$,
$\Gamma_2 = \{(x,y) : x = \pm 1,\ -2 \le y \le -1 \text{ or } 1 \le y \le 2;\ y = \pm 2,\ -1 \le x \le 1\}$,
$\gamma_1 = \{(x,y) : y = \pm 1,\ -1 \le x \le -1\}$, $\gamma_2 = \{(x,y) : x = \pm 1,\ -1 \le y \le 1\}$.

3. The algorithm. The algorithm consists of two parts.

3.1. First part - the Schwartz method. The Schwartz iterative method consists of two A and B problems. Problem A:

$$\Delta u_{2k-1}(x,y) = f(x,y), \quad (x,y) \in \Omega_1, \tag{3}$$

$$u_{2k-1}(x,y) = g_{2k-1}(x,y), \quad (x,y) \in \Gamma_1 \cup \gamma_1, \tag{4}$$

where

$$g_{2k-1}(x,y) = \begin{cases} g(x,y), & (x,y) \in \Gamma_1, \\ u_{2k-2}(x,y), & (x,y) \in \gamma_1. \end{cases} \tag{5}$$

Problem B:

$$\Delta u_{2k}(x,y) = f(x,y), \quad (x,y) \in \Omega_2, \tag{6}$$

$$u_{2k}(x,y) = g_{2k}(x,y), \quad (x,y) \in \Gamma_2 \cup \gamma_2, \tag{7}$$

where

$$g_{2k}(x,y) = \begin{cases} g(x,y), & (x,y) \in \Gamma_2, \\ u_{2k-1}(x,y), & (x,y) \in \gamma_2, \end{cases} \tag{8}$$

$u_{2k-1}(x,y)$ and $u_{2k}(x,y)$ are unknown functions,$k = 1,2,3,\cdots,k_{max}$, k_{max} is a maximal number of iteratives. We take $u_0(x,y) \equiv 0$ as a given initial approximation. In this case

$$g_1(x,y) = \begin{cases} g(x,y), & (x,y) \in \Gamma_1, \\ 0, & (x,y) \in \gamma_1. \end{cases}$$

and $g_1(x,y)$ for the continuity of boundary data we need

$$g(-1,-1) = g(-1,1) = g(1,-1) = g(1,1) = 0.$$

3.2. Second part - the Galerkin method. Let's expand unknown $u_{2k-1}(x,y)$ and $u_{2k}(x,y)$ functions as the Fourier-Legendre finite series

$$u_{2k-1}(x,y) = \sum_{i,j=1}^{N} u_{2k-1}^{ij} \varphi_i(x/2)\varphi_j(y) + G_{2k-1}(x,y), \tag{9}$$

$$u_{2k}(x,y) = \sum_{i,j=1}^{N} u_{2k}^{ij} \varphi_i(x)\varphi_j(y/2) + G_{2k}(x,y), \tag{10}$$

where each u_{2k-1}^{ij} and u_{2k}^{ij} N^2 are unknown coefficients, which are found by the Galerkin method;

$$G_{2k-1}(x,y) = g_{2k-1}(x,y), \quad (x,y) \in \Gamma_1 \cup \gamma_1 \tag{11}$$

$$G_{2k}(x,y) = g_{2k}(x,y), \quad (x,y) \in \Gamma_2 \cup \gamma_2 \tag{12}$$

With the aim to satisfy the boundary conditions (4), (7) the functions $G_{2k-1}(x,y)$ and $G_{2k}(x,y)$ must have the following forms:

$$G_{2k-1}(x,y) = \frac{1-y}{2} g_{2k-1}(x,-1) + \frac{1+y}{2} g_{2k-1}(x,1), \quad \text{when } y = \pm 1,$$

or

$$\frac{2-x}{4} g_{2k-1}(-2,y) + \frac{2+x}{4} g_{2k-1}(2,y), \quad \text{when } x = \pm 2;$$

$$G_{2k}(x,y) = \frac{1-x}{2} g_{2k}(-1,y) + \frac{1+x}{2} g_{2k}(1,y), \quad \text{when } x = \pm 1.$$

or

$$\frac{2-y}{4}g_{2k}(x,-2)+\frac{2+y}{4}g_{2k}(x,2)),\quad \text{when } y=\pm 2;$$

In order to find the unknown coefficients u_{2k-1}^{ij} and u_{2k}^{ij} let's apply the Galerkin method:

$$(\Delta u_{2k-1}(x,y)-f(x,y),\ \varphi_l(x/2)\varphi_m(y))=0,\quad l,m,=1,2,\cdots,N, \tag{13}$$

$$(\Delta u_{2k}(x,y)-f(x,y),\ \varphi_l(x)\varphi_m(y/2))=0,\quad l,m,=1,2,\cdots,N, \tag{14}$$

where $\varphi_i(t)=P_{i+1}(t)-P_{i-1}(t)$, $P_i(t)$ are Legendre polynomial. Using basic properties of Legendre polynomial we have

$$\begin{aligned}&\sum_{i=1}^{N}\sum_{j=1}^{N}u_{2k-1}^{ij}\left(\frac{\partial^2}{\partial x^2}\varphi_i\left(\frac{x}{2}\right),\varphi_l\left(\frac{x}{2}\right)\right)\cdot(\varphi_j(y),\varphi_m(y))\\&+\sum_{i=1}^{N}\sum_{j=1}^{N}u_{2k-1}^{ij}\left(\varphi_i\left(\frac{x}{2}\right),\varphi_l\left(\frac{x}{2}\right)\right)\cdot\left(\frac{\partial^2}{\partial y^2}\varphi_j(y),\varphi_m(y)\right)=F1_{lm},\end{aligned}\tag{15}$$

$$\begin{aligned}&\sum_{i=1}^{N}\sum_{j=1}^{N}u_{2k}^{ij}\left(\frac{\partial^2}{\partial x^2}\varphi_l(x),\varphi_l(x)\right)\cdot\left(\varphi_j\left(\frac{y}{2}\right),\varphi_m\left(\frac{y}{2}\right)\right)\\&+\sum_{i=1}^{N}\sum_{j=1}^{N}u_{2k}^{ij}\left(\varphi_i(x),\varphi_l(x)\right)\cdot\left(\frac{\partial^2}{\partial y^2}\varphi_j\left(\frac{y}{2}\right),\varphi_m\left(\frac{y}{2}\right)\right)=F2_{lm},\end{aligned}\tag{16}$$

where

$$F1_{lm}=-\left(\Delta G_{2k-1}(x,y)-f(x,y),\quad \varphi_l\left(\frac{x}{2}\right)\varphi_m(y)\right),\quad l,m=1,2,\cdots,N.$$

$$F2_{lm}=-\left(\Delta G_{2k}(x,y)-f(x,y),\quad \varphi_l(x)\varphi_m\left(\frac{y}{2}\right)\right),\quad l,m=1,2,\cdots,N.$$

4. Numerical experiments and results of calculation. Let's consider a particular case for N = 5. If $N<5$, then the matrix of the system of equations will not be five-pointed. System's matrix M is the same for A, as well as in B cases, only the right sides are different

$$M=\begin{pmatrix}A_1&0&B_1&0&0\\0&A_2&0&B_2&0\\C_3&0&A_3&0&B_3\\0&C_4&0&A_4&0\\0&0&C_5&0&A_5\end{pmatrix}$$

where $A_i(i=\overline{1,5})$, are three pointed ribbon matrixes, $B_i(i=\overline{1,3})$, $C_i(i=\overline{3,5})$ the diagonal matrixes, O -is a zero matrix. The program code is examined for the following test problem:

$$\Delta u(x,y)=f(x,y),\quad (x,y)\in\Omega, \tag{17}$$

$$u(x,y)=0,\quad (x,y)\in\partial\Omega, \tag{18}$$

where the right side is

$$f(x,y) = (y^2-1)(y^2-4)(12x^2-10) + (x^2-1)(x^2-4)(12y^2-10),$$

and the exact solution is

$$u(x,y) = (x^2-1)(x^2-4)(y^2-1)(y^2-4).$$

There is represented an exact, as well as an approximate solution's values in the origin for the different number of iterations in table 1.

$u(0,0)$	$u_1(0,0)$	$u_2(0,0)$	$u_3(0,0)$	$u_4(0,0)$	$u_5(0,0)$	$u_6(0,0)$
16	10.7341	11.4653	12.3942	13.4589	15.0124	15.3827

Table 1.

As we use Schwartz's algorithm, we can take $u_{2k-1}(x,y)$, as well as $u_{2k}(x,y)$ function, as an approximate solution for a problem (1), (2) in the intersection of the rectangles Ω_1 and Ω_2. The numerical experiment made an a testing example, shows a convergence of Schwartz's algorithm. A counting process is stable, as corresponding matrix of algebraic equation system has diagonal dominating property relative to rows.

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