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PARTIALLY UNKNOWN BOUNDARY AXIS SYMMETRIC PROBLEM OF PLANE ELASTICITY THEORY

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Abstract. This paper deals with a problem of plane elasticity theory for a doubly connected body whose external boundary is a rhombus boundary and the internal boundary is a required full-strength hole. Absolutely smooth stamps with rectilinear bases are applied to the linear parts of the boundary, and the middle points of these stamps are under the action of concentrated forces. The hole boundary is subject to uniform pressure. Using the methods of complex analysis, the equations of an unknown part of the boundary are determined.

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The axis-symmetric and cycle symmetric problems of the plane theory of elasticity and plate bending with partially unknown boundaries are studied in [1], [2], [3], [5], [6] and [7]. In this article, the axially symmetric problem of plane elasticity theory for rhombus weakened with full-strength hole is considered. Formulas of Kolovos-Muskhelishvili are used for investigating this problem. The solution is written in quadratures.

Let an isotropic body on the plane $z = x + iy$ occupies a double connected domain S, whose external boundary is a rhombus boundary, whose diagonals lie at the coordinate axes OX and OY, the internal boundary is a required full-strength hole, whose symmetric axes are the rhombus diagonals.

Let to every link of the broken line be applied absolutely smooth rigid stamps with rectilinear bases, and let the middle points of these stamps be under the action of concentrated forces P. The unknown full-strength contour is subject to uniform pressure Q. Under the above assumptions, the tangential stresses $\tau_{ns} = 0$ are zero along the entire boundary of the domain S and the normal displacements of every link of external boundary $v_n = v = const.$

Consider the following problem : Find the shape of the unknown hole and the stress state of the given body such that the tangential normal stress arising at it would take constant value.

Since the problem is axially symmetric, then to investigate the stated problem, it is sufficient to consider the curvilinear quadrangle $A_1A_2A_3A_4$ which will be denoted by D. The normal displacements v_n and the tangential stresses τ_{ns} are equal to zero at each segment A_1A_2, A_3A_4.

Let us introduce the following notations $\Gamma_1 = A_1A_2$, $\Gamma_2 = A_2A_3$, $\Gamma_3 = A_3A_4$, $\gamma = A_4A_1$, $\Gamma = \bigcup\limits_{j=1}^{3} \Gamma_j$, $P_1 = \int\limits_{\Gamma_1} \sigma_n ds$, $P_2 = \int\limits_{\Gamma_2} \sigma_n ds$, $P_3 = \int\limits_{\Gamma_3} \sigma_n ds, \sigma_n$-is the normal stress,

$$v_n = \begin{cases} 0, & t \in \Gamma_3 \cup \Gamma_1 \\ v & t \in \Gamma_2 \end{cases}, \tag{1}$$

$$\tau_{ns} = 0, \quad t \in \Gamma \cup \gamma, \tag{2}$$

$$\sigma_n = -Q, \sigma_s = K, \quad t \in \gamma \tag{3}$$

$$P_2 = -P, \quad P_1 = -P\cos\beta - Q, \quad P_3 = -P\sin\beta - Q, \tag{4}$$

$$\beta = \angle A_1 A_2 A_3.$$

Let the points A_1, A_2, A_3, A_4 be counted in a positive direction and its affixes be denoted by the same symbols. Assume that A_1 is the origin of the broken line Γ.

On the basis of the well-known Kolosov-Muskelishvili's formulas [3], the problem reduces to finding the functions ψ, φ which are holomophic in the domain D with the following conditions

$$\mathrm{Re} e^{-i\alpha(t)} \left(\chi\varphi(t) - t\overline{\varphi'(t)} - \overline{\psi(t)} \right) = 2\mu\nu_n(t), \quad t \in \Gamma, \tag{5}$$

$$\mathrm{Re} e^{-i\alpha(t)} \left(\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi\,(t)} \right) = c\,(t)\,, \quad t \in \Gamma, \tag{6}$$

$$\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)} = 0, \quad t \in \gamma, \tag{7}$$

$$\mathrm{Re}\varphi'(t) = \frac{\sigma_n + \sigma_s}{4} = \frac{K + Q}{4}, \quad t \in \gamma, \tag{8}$$

where - χ, μ are elasticity constants, c(t) is a piecewise – constant function, $\alpha(t)$ - is the angle formed between the external normal n to contour and the abscissa axis Ox, $\alpha(t) = \alpha_k, \in \Gamma_k, k = 1,2,3,$ $\alpha_1 = -\frac{\pi}{2}$, $\alpha_2 = \frac{\pi}{2} - \beta$, $\alpha_3 = \pi$, $c(t) = \mathrm{Re}\left(e^{-i\alpha(t)} i \left(\int\limits_{A_1}^{t} \sigma_n(s_0) e^{i\alpha(s_0)} ds_0 \right) \right)$. It is proved, that one potential is a linear function $\varphi\,(z) = \frac{K+Q}{4} z$. Substituting the value $\varphi\,(z)$, $c(t)$ into the boundary conditions (6), (7) one gets the following problem

$$\mathrm{Re}\left[e^{-i\alpha(t)} \left(\frac{K+Q}{2} t + \overline{\psi\,(t)} \right) \right] = c(t), \tag{9}$$

$$\mathrm{Re}\, t e^{-i\alpha(t)} = \mathrm{Re} e^{-i\alpha(t)} A(t), \quad t \in \Gamma, \tag{10}$$

$$\frac{K+Q}{2} t + \overline{\psi\,(t)} = 0, t \in \gamma. \tag{11}$$

Let the function $z = \omega\,(\varsigma)$, $\varsigma = \xi + i\eta$ map the semicircle $|\varsigma| < 1, \mathrm{Im}\varsigma > 0$ conformally onto the domain D. It is assumed that the vertices A_k of rhombus line correspond to the points a_k of the semicircle $|\varsigma| = 1$, $\mathrm{Im}\varsigma > 0$, $a_k = \omega^{-1}(A_k)$, $k = 1,2,3,4$. It is assumed, that $a_1 = 1, a_4 = -1, a_3 = i$. Here we can fix three points and remaining ones are to be defined. Then the diameter $-1 \le \xi \le 1$ is mapped onto arc

A_4A_1 and the semi-circumference $\gamma_0 : |\gamma_0| = 1, \text{Im}\varsigma > 0$ is mapped onto the broken line Γ. The point $a_2 = e^{i\theta_2}$, $0 < \theta_2 < \frac{\pi}{2}$ is to be defined. Hence, by virtue of (9), (10), (11) for functions $\psi_0(\varsigma) = \psi(\omega(\varsigma))$ and one obtains:

$$\text{Re}e^{-i\alpha(\sigma)}\overline{\psi_0(\sigma)} = -\frac{K+Q}{2}\text{Re}e^{-i\alpha(\sigma)}A(\sigma) + c(\sigma), \quad \sigma \in \gamma_0, \tag{12}$$

$$\text{Re}e^{-i\alpha(\sigma)}\omega(\sigma) = \text{Re}e^{-i\alpha(\sigma)}A(\sigma), \quad \sigma \in \gamma_0, \tag{13}$$

$$\frac{K+Q}{2}\omega(\sigma) + \overline{\psi_0(\sigma)} = 0, \quad \gamma \in \gamma(-1,1). \tag{14}$$

Consider the new unknown function $W(\varsigma)$ defined by

$$W(\varsigma) = \begin{cases} \dfrac{K+Q}{2}\omega(\varsigma), & |\varsigma| < 1, \ \text{Im}\varsigma > 0, \\ -\overline{\psi_0(\overline{\varsigma})}, & |\varsigma| < 1, \ \text{Im}\varsigma < 0. \end{cases} \tag{15}$$

By virtue of (14) it is easy to verify that $W(\varsigma)$ –is a holomorphic function into the circle $|\varsigma| < 1$.

By virtue of (12) - (13) the function defined by (15) satisfies the boundary condition, $\gamma' = \gamma_0 \cup \gamma_0^*$, $\alpha(\sigma) = \alpha(\overline{\sigma})$:

$$\text{Re}e^{-i\alpha(\sigma)}W(\sigma) = f(\sigma), \quad \sigma \in \gamma'. \tag{16}$$

Thus, the problem in question has been reduced to the Riemann–Hilbert problem with piecewise-constant coefficients. Here the problem is reduced to the Dirichlet problem for a circle and its solution is presented by Schwarz formula, which is computationally convenient.

We will have the form

$$W(\varsigma) = \frac{\varsigma X(\varsigma)}{\pi i}\int_{\gamma'} \frac{f(\sigma)e^{i\alpha(\sigma)}d\varsigma}{X(\sigma)\sigma(\sigma-\varsigma)}, \tag{17}$$

where $X(\varsigma)$ is presented, as

$$X(\varsigma) = \frac{(\varsigma - a_2)^{\frac{\alpha_1-\alpha_2}{\pi}+1}(\varsigma - a_3)^{\frac{\alpha_2-\alpha_3}{\pi}+1}(\varsigma - \overline{a}_3)^{\frac{\alpha_3-\alpha_2}{\pi}}(\varsigma - \overline{a}_2)^{\frac{\alpha_2-\alpha_1}{\pi}}}{\varsigma}\sqrt{\overline{a}_2\overline{a}_3}. \tag{18}$$

By virtue of (15), the equation of the contour $z = \omega(\xi)$ is presented by

$$\omega(\xi) = \frac{2W(\xi)}{K+Q}, \quad -1 < \xi < 1$$

From relationship (15) one can define the functions ω, ψ. Thus the equation of the contour and stress state of body is defined. Thus, the part of the required full-strength contour is constructed by $z = \omega(\xi)$. Since the problem is axially symmetric, then other parts of this graphic is obtained by its axis symmetric mapping with respect to coordinate axes Ox and Oy.

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