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# A NUMERICAL METHOD OF THE ONE-DIMENSIONAL NONLINEAR REISSNER SYSTEM

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**Abstract**. It is proved that the linear problem of static deformation of the Reissner plate is solvable and the approximate process is convergent.

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**AMS subject classification**: 35L70, 65M60.

In [1] it is noted that the construction of the theory of boundary value problems and the substantiation of approximate methods for the Reissner model, which takes into account both geometrical nonlinearity and shear stresses, is one of the unsolved mathematical problems of plate and shell theory. The question of the solvability of two-dimensional variants of the Reissner system has not been investigated because of the complex nature of nonlinearity.

The present paper considers a three-layer plate which is uniformly loaded with respect to the width. This assumption enables us to discard one variable together with concomitant unknown functions in the system of equations which in [2] is written in the divergent form.

A method is proposed here for the solution of a boundary value problem, which consists in reducing the initial system to some integro-differential system of equations. This technique makes it possible to find the required a priori estimates. In the discussion of this issue, we apply the method of proof used in [3] and [4].

Let us assume that the three-layer plate is homogeneous and has the facial layers of the same thickness.

The deformation of the plate under the above assumptions can be described by the following system of ordinary differential equations

$$N' = 0, \tag{1.1}$$

$$M' - Q = 0, \tag{1.2}$$

$$[(N + M)w' + Q]' - Qw' = -q, \tag{1.3}$$

where

$$N = \frac{2Eh_1}{1-\nu^2}\left[u' + \frac{1}{2}\,(w')^2\right], \quad M = \frac{2Eh_1(h_1+h_2)^2}{2(1-\nu^2)}\,\beta', \quad Q = (h_1+h_2)G(\beta + w').$$

The sought functions are $u = u(x)$, $\beta = \beta(x)$ and $w = w(x)$, where $u$ and $w$ are the displacements of the midsurface point along $x$ and $z$, and $\beta$ is the rotation angle of the normal in the $xz$-plane. The given function $q = q(x)$ corresponds to the transverse load, $x$ is a spatial load, $0 \le x \le l$; $h_1, h_2$ are the thicknesses of the middle and the outer layers; $G$, $E$ are the elasticity and rigidity moduli, $\nu$ is Poissons ratio, $0 < \nu < 0.5$.

We complement (1.1)-(1.3) with the following boundary conditions

$$u(0) = u(l) = 0, \quad w(0) = w(l) = 0, \quad M(0) = M(l) = 0, \tag{2}$$

Let us transform system (1.1)-(1.3). Using (1.1), twice integrating and taking into account the boundary condition for $u$, we obtain

$$u = -\frac{1}{2}\int_0^x (w'(\xi))^2\,dx + \frac{x}{2l}\int_0^l (w'(x))^2\,dx. \tag{3}$$

Now let us express the function $\beta$ in terms of $w$. To this end, we consider the problem

$$v'' - k^2 v = f, \tag{4}$$

$$v'(0) = v'(l) = 0. \tag{5}$$

We write a solution of (4), (5) in the form $v = v_1 + v_2$, where $v_1$ satisfies (4) and the condition $v_1(0) = v_1(l) = 0$, while $v_2$ is a particular solution of equation (4) for $f = 0$, such that (5) is fulfilled on the boundary when $v_1 + v_2$. $v_1$ can be constructed by means of the Green function using the formula

$$v_1(x) = \int_0^l G(x,\xi) f(\xi)\,dx,$$

$$G(x,\xi) = \begin{cases} -\dfrac{\sinh(\alpha\xi)\sinh[\alpha(l-x)]}{\alpha\sinh(\alpha l)} & \text{for} \quad \xi \le x, \\ -\dfrac{\sinh(\alpha x)\sinh[\alpha(l-\xi)]}{\alpha\sinh(\alpha l)} & \text{for} \quad \xi > x. \end{cases}$$

As to the summand $v_2$, it is equal to $C_1 l^{kx} + C_2 l^{-kx}$, where $C_1$ and $C_2$ are the uniquely defined constants.

Let us apply the reasoning, which we have used above for (4), (5), to (1.2) and the condition $\beta'(0) = \beta'(l) = 0$ implied by (2). As a result we come to the equality

$$\beta = \frac{\alpha^2}{\sinh\alpha l}\left[\cosh\alpha(l-x)\int_0^x w(\xi)\sinh\alpha\xi\,d\xi - \cosh\alpha x\int_x^l w(\xi)\sinh\alpha(l-\xi)\,d\xi\right], \tag{6}$$

where

$$\alpha^2 = \frac{2G(1-\nu^2)}{Eh_1(h_1+h_2)}, \quad \alpha > 0.$$

Using (3) and (6) in (1.3), we obtain

$$\left[\gamma_0 + \gamma_1\int_0^l (w')^2 dx + \gamma(x,w)\right] w'' + \alpha^2\gamma(x,w) = -q, \tag{7}$$

where

$$\gamma_0 = (h_1+h_2)G, \quad \gamma_1 = \frac{Eh_1}{l(1-\nu^2)},$$

$$\gamma(x,w) = \gamma_0 w - \gamma_2\left\{\sinh[\alpha(l-x)]\int_0^x w(\xi)\sinh(\alpha\xi)\,d\xi + \sinh(\alpha x)\int_x^l w(\xi)\sinh[\alpha(l-\xi)]\,d\xi\right\}, \quad \gamma_2 = \frac{\alpha\gamma_0}{\sinh(\alpha l)}.$$

From (2) we take the condition

$$w(0) = w(l) = 0. \tag{8}$$

(7), (8) form an independent problem for $w$.

**Theorem.** *Let the conditions*

$$q \in L_2(0,1)$$

*and the inequality*

$$\frac{(h_1+h_2)^3 l^2 (1-\gamma^2)}{128 E h_1} < 1.$$

*be fulfilled. Then there exists a solution $w \in \overset{\circ}{W}{}_2^1(0,l) \cap W_2^2(0,l)$ of problem (7),(8).*

An approximate solution of problem (7),(8) can be found by the Bubnov-Galerkin method. The sequence of approximate solutions converges in a weak sense to $w$.

**Proof.** In proving the theorem, the following notation is used: $E_n$ is the Euclidean space of $n$-dimensional vectors with a scalar product denoted by $(\cdot,\cdot)_0$, $(\cdot,\cdot)$ is a scalar product in $L_2(0,l)$, $|\cdot|_C$ is a norm in the space $C(0,l)$. We define

$$|v|_{\overset{\circ}{W}{}_2^1} = \left( \int_0^l (v')^2 dx \right)^{\frac{1}{2}}, \quad |v|_{\overset{\circ}{W}{}_2^1 \cap W_2^2} = \left( \int_0^l (v'')^2 dx \right)^{\frac{1}{2}}.$$

For brevity, these norms will be denoted by $|\cdot|_1$ and $|\cdot|_2$, respectively.

Let us rewrite equation (7) in the operator form

$$\Phi(w) = 0,$$

where

$$\Phi(w) = \left[ \gamma_0 + \gamma_1 \int_0^l (w'(\xi))^2 d\xi + \gamma(x,w) \right] w'' + \alpha^2 \gamma(x,w) + q,$$

$$\gamma_0 = (h_1+h_2)G, \quad \gamma_1 = \frac{E h_1}{(1-\nu^2) l},$$

$$\gamma(x,w) = \gamma_0 w - \gamma_2 \Big\{ \sinh[\alpha(l-x)] \int_0^x w(\xi) \sinh(\alpha \xi)\, d\xi + \sinh(\alpha x) \int_x^l w(\xi) \sinh[\alpha(l-\xi)]\, d\xi \Big\}, \quad \gamma_2 = \frac{\alpha \gamma_0}{\sinh(\alpha l)}.$$

A function $w \in \overset{\circ}{W}{}_2^1(0,l) \cap W_2^2(0,l)$ that satisfies the equality

$$(\Phi(w), \varphi) = 0 \quad \forall \varphi \in \overset{\circ}{W}{}_2^1(0,l) \cap W_2^1(0,l).$$

is called a generalized solution of problem (7),(8).

A solution of problem (7),(8) is sought by the Bubnov-Galerkin method. We construct the sequence of approximations $\{w_n\}$, $n = 1, 2, \ldots$, of the form

$$w_n = \sum_{i=1}^n w_{n_i} \nu_i, \quad \nu_i = \sin \frac{i \pi x}{l},$$

where the coefficients $w_{n_i}$ are defined from the finite system

$$(\Phi(w_n), \nu_i) \equiv ([\gamma_0 + \gamma_1 |w_n|_1^2 + \gamma(x, w_n)] w_n', \nu_i') \\ + (\gamma_x'(x, w_n) \cdot w_n' - \alpha^2 \gamma(x, w_n) - q, \nu_i) = 0. \quad (9)$$

We prove that system (9) is solvable and derive the estimate $|w_n|_2 \leq const$.

The obtained a priori estimate allows us to perform passage to the limit, and thereby to complete the proof of the theorem.

Using $w$ found by means of (3) and (6), we can construct $u$ and $\beta$.

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