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SOME PROPERTIES OF SOLUTION AND VARIABLE DIRECTIONS DIFFERENCE SCHEME FOR ONE SYSTEM OF NONLINEAR THREE-DIMENSIONAL PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. One three-dimensional system of nonlinear partial differential equations is considered. Some properties of solution is given. Difference scheme of variable directions is constructed.

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In the cylinder $\bar{\Omega} \times [0, T]$ consider the following nonlinear system:

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x_1}\left(V_1 \frac{\partial U}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(V_2 \frac{\partial U}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(V_3 \frac{\partial U}{\partial x_3}\right), \tag{1}$$

$$\frac{\partial V_\alpha}{\partial t} = -V_\alpha + g_\alpha\left(V_\alpha \frac{\partial U}{\partial x_\alpha}\right), \tag{2}$$

with initial and boundary conditions

$$U(x,0) = U_0(x), \quad V_\alpha(x,0) = V_{\alpha 0}(x), \quad \alpha = 1,2,3, \quad x \in \bar{\Omega}, \tag{3}$$

$$U(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,T]. \tag{4}$$

Here $\Omega = \{x = (x_1, x_2, x_3) : \ 0 < x_1 < 1, \ 0 < x_2 < 1, \ 0 < x_3 < 1\}$, $\partial\Omega$ is the boundary of the domain Ω, T is some fixed positive number, U_0, $V_{\alpha 0}$, g_α are given sufficiently smooth functions, such that:

$$V_{\alpha 0}(x) \geq \delta_0, \quad x \in \bar{\Omega}, \tag{5}$$

$$\gamma_0 \leq g_\alpha(\xi_\alpha) \leq G_0, \quad |g'_\alpha(\xi_\alpha)| \leq G_1, \quad \xi_\alpha \in R, \quad \alpha = 1,2,3, \tag{6}$$

where δ_0, γ_0, G_0, G_1 are some positive constants.

In two-dimensional case (1), (2) system describes the vein formation in meristematic tissues of young leaves [1]. In [1], [2] some qualitative and structural properties of the solutions of the initial-boundary value problems for the (1), (2) type system are established. In [2] investigations are carried out for one-dimensional analog of the (1), (2). Many scientific works are devoted to construction of numerical resolution algorithms for (1), (2) type models [3]-[13].

Naturally arises the question of constructing the economical algorithms for solution of multi-dimensional problem (see, for example, [14] and references therein).

In the present work, based on work [15], one kind of such algorithm for problem (1)-(6) is given. Analogical investigations for (1), (2) type systems are carried out in [6]-[9], [11], [12]. In [3] average model of sum approximation is studied as well.

Suppose that all necessary consistence conditions are satisfied and there exists the sufficiently smooth solution of the problem (1)-(4).

Under the conditions (5), (6) from (2)-(4) we have

$$V_\alpha(x,t) = e^{-t}V_{\alpha 0}(x) + e^{-t}\int_0^t e^\tau g_\alpha\left(V_\alpha \frac{\partial U}{\partial x_\alpha}\right)d\tau \geq e^{-t}\delta_0$$
$$+e^{-t}\int_0^t e^\tau \gamma_0 d\tau = \sigma_0 = const > 0, \quad (x,t)\in \bar{Q} = \bar{\Omega}\times[0,T], \quad \alpha = 1,2,3. \tag{7}$$

Analogically we prove the upper boundedness of the functions $V_\alpha(x,t)$

$$V_\alpha(x,t) \leq \Delta_0 = const, \quad (x,t)\in\bar{Q} \tag{8}$$

and at last using (2), (8) – the estimations

$$\left|\frac{\partial V_\alpha}{\partial t}\right| \leq C, \quad \alpha = 1,2,3, \quad (x,t)\in \bar{Q}. \tag{9}$$

Here C is a positive constant.

Later we shall follow known notations:

$$\bar{\omega}_h = \{x_{i_1i_2i_3} = (i_1h_1, i_2h_2, i_3h_3), \quad i_\beta = 0,...,N_\beta, \quad N_\beta h_\beta = 1, \quad \beta = 1,2,3\},$$
$$\bar{\omega}_{2h} = \{x_{i_1i_2i_3} = (i_1h_1,\ (i_2-1/2)\,h_2,\ i_3h_3),\ i_1 = 0,...,N_1,\ i_2 = 1,...,N_2,\ i_3 = 0,...,N_3\},$$
$$\bar{\omega}_{3h} = \{x_{i_1i_2i_3} = (i_1h_1,\ i_2h_2,\ (i_3-1/2)\,h_3),\ i_1 = 0,...,N_1,\ i_2 = 0,...,N_2,\ i_3 = 1,...,N_3\},$$
$$\bar{\omega}_{1h} = \{x_{i_1i_2i_3} = ((i_1-1/2)\,h_1,\ i_2h_2,\ i_3h_3),\ i_1 = 1,...,N_1,\ i_2 = 0,...,N_2,\ i_3 = 0,...,N_3\},$$
$$\omega_h = \Omega\cap\bar{\omega}_h, \quad \gamma_h = \bar{\omega}_h\backslash\omega_h, \quad \bar{\omega}_h = \omega_h\cup\gamma_h, \omega_\tau = \{t_j = j\tau, \quad j = 0,...,j^*, \quad j^*\tau = T\},$$
$$\bar{\omega}_{h\tau} = \bar{\omega}_h\times\omega_\tau, \quad \bar{\omega}_{\alpha h\tau} = \bar{\omega}_{\alpha h}\times\omega_\tau, \quad \alpha = 1,2,3.$$

Here h_α is the space step in direction x_α and τ is the time step on the interval $[0,T]$.

Define the following inner products and the norms for the discrete functions y and z given on $\bar{\omega}_h$:

$$(y,z]_1 = \sum_{i_1=1}^{N_1}\sum_{i_2=1}^{N_2-1}\sum_{i_3=1}^{N_3-1} y_{i_1i_2i_3}z_{i_1i_2i_3}h_1h_2h_3, \quad (y,z]_2 = \sum_{i_1=1}^{N_1-1}\sum_{i_2=1}^{N_2}\sum_{i_3=1}^{N_3-1} y_{i_1i_2i_3}z_{i_1i_2i_3}h_1h_2h_3,$$
$$(y,z]_3 = \sum_{i_1=1}^{N_1-1}\sum_{i_2=1}^{N_2-1}\sum_{i_3=1}^{N_3} y_{i_1i_2i_3}z_{i_1i_2i_3}h_1h_2h_3, \quad \|y]|_\alpha = (y,y]_\alpha^{1/2}, \quad \alpha = 1,2,3.$$

The corresponding inner products and the norms on $\bar{\omega}_{\alpha h}$ can be defined in a similar way. Introduce also the following known notations:

$$y = y^j_{i_1i_2i_3} = y(x_{i_1i_2i_3}, t_j), \quad \hat{y} = y^{j+1}_{i_1i_2i_3} = y(x_{i_1i_2i_3}, t_{j+1}), \quad y_t = \frac{\hat{y}-y}{\tau},$$
$$y_{\bar{x}_1} = \frac{y^j_{i_1i_2i_3} - y^j_{i_1-1,i_2i_3}}{h_1} \quad y_{x_1} = \frac{y^j_{i_1+1,i_2i_3} - y^j_{i_1i_2i_3}}{h_1},$$
$$y_{\bar{x}_1x_1} = \frac{y^j_{i_1+1,i_2...i_3} - 2y^j_{i_1i_2i_3} + y^j_{i_1-1,i_2i_3}}{h_1^2}.$$

The difference quotients for the discrete functions $y_{\bar{x}_2}$, y_{x_2}, $y_{\bar{x}_3}$, y_{x_3}, $y_{\bar{x}_2 x_2}$, $y_{\bar{x}_3 x_3}$ and for the discrete functions given on $\bar{\omega}_{\alpha h\tau}$ are defined similarly.

Let us correspond to the problem (1)-(4) the following variable directions type difference scheme:

$$u_{1t} = \left(\hat{v}_\beta \hat{u}_{1\bar{x}_1}\right)_{x_1} + \left(v_2 u_{2\bar{x}_2}\right)_{x_2} + \left(v_3 u_{3\bar{x}_3}\right)_{x_3}, \tag{10}$$

$$u_{2t} = \left(\hat{v}_1 \hat{u}_{1\bar{x}_1}\right)_{x_1} + \left(\hat{v}_2 \hat{u}_{2\bar{x}_2}\right)_{x_2} + \left(v_3 u_{3\bar{x}_3}\right)_{x_3}, \tag{11}$$

$$u_{3t} = \left(\hat{v}_1 \hat{u}_{1\bar{x}_1}\right)_{x_1} + \left(\hat{v}_2 \hat{u}_{2\bar{x}_2}\right)_{x_2} + \left(\hat{v}_3 \hat{u}_{3\bar{x}_3}\right)_{x_3}, \tag{12}$$

$$v_{\alpha t} = -\hat{v}_\alpha + g_\alpha \left(v_\alpha u_{\alpha\bar{x}_\alpha}\right), \tag{13}$$

$$u_\alpha(x,0) = U_0(x), \quad x \in \bar{\omega}_h, \tag{14}$$

$$v_\alpha(x,0) = v_{\alpha 0}(x), \quad x \in \bar{\omega}_{1h}, \tag{15}$$

$$u_\alpha(x,t) = 0, \quad (x,t) \in \gamma_h \times \omega_\tau, \quad \alpha = 1,2,3. \tag{16}$$

In (10)-(12) the discrete functions u_α are defined on $\bar{\omega}_{h\tau}$ and v_α – on $\bar{\omega}_{\alpha h\tau}$ respectively.

For the exact solution U, V_1, V_2, V_3 of the problem (1)-(4) we have:

$$\begin{gathered}
U_t = \left(\hat{V}_\beta \hat{U}_{1\bar{x}_1}\right)_{x_1} + \left(V_2 U_{2\bar{x}_2}\right)_{x_2} + \left(V_3 U_{3\bar{x}_3}\right)_{x_3} + \varphi_1, \\
U_t = \left(\hat{V}_1 \hat{U}_{1\bar{x}_1}\right)_{x_1} + \left(\hat{V}_2 \hat{U}_{2\bar{x}_2}\right)_{x_2} + \left(V_3 U_{3\bar{x}_3}\right)_{x_3} + \varphi_2, \\
U_t = \left(\hat{V}_1 \hat{U}_{1\bar{x}_1}\right)_{x_1} + \left(\hat{V}_2 \hat{U}_{2\bar{x}_2}\right)_{x_2} + \left(\hat{V}_3 \hat{U}_{3\bar{x}_3}\right)_{x_3} + \varphi_3, \\
V_{\alpha t} = -\hat{V}_\alpha + g_\alpha \left(V_\alpha U_{\alpha\bar{x}_\alpha}\right) + \psi_\alpha, \\
U(x,0) = U_0(x), \quad x \in \bar{\omega}_h, \quad V_\alpha(x,0) = V_{\alpha 0}(x), \quad x \in \bar{\omega}_{\alpha h}, \quad \alpha = 1,2,3, \\
U(x,t) = 0, \quad (x,t) \in \gamma_h \times \omega_\tau.
\end{gathered}$$

It is clear that each of the difference equations (10) - (13) approximate the corresponding differential equations (1), (2). Under the sufficient smoothness of the exact solution U, V_1, V_2, V_3 the approximate errors φ_α and ψ_α are the values of order $O\left(\tau + h_1^2 + h_2^2 + h_3^2\right)$ and $O\left(\tau + h_\alpha^2\right)$ respectively.

The theorems of absolute stability and convergence of (10)-(16) type variable direction scheme are shown in [4]. According to [3] the averaged model of sum approximation are constructed as well. The test numerical experiments are carried out by using these schemes. Comparative analysis of numerical results is carried out.

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