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ON THE 2-D NONLINEAR SYSTEMS OF EQUATIONS FOR NON-SHALLOW SHELLS

Meunargia T.

Abstract. I. Vekua has constructed several versions of the refined theory of thin and shallow shells. Using the reduction methods of I. Vekua, the 2-D system of equations for geometrically and physically nonlinear theory of non-shallow shells is obtained.

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A complete system of equilibrium equation and the stress-strain relations of the 3-D geometrically and physically nonlinear theory can be written as:

$$\hat{\nabla}_i \boldsymbol{\tau}^i + \boldsymbol{\Phi} = 0, \quad \boldsymbol{\tau}^i = (E^{ijpq} + E^{ijpqks} e_{ks}) e_{pq} (\boldsymbol{R}_j + \partial_j \boldsymbol{U}), \quad (i,j,p,q,k,s = 1,2,3) \tag{1}$$

where $\hat{\nabla}_i$ are covariant derivatives with respect to the space curvilinear coordinates x^i, $\boldsymbol{\tau}^i$ and $\boldsymbol{\Phi}$ are the contravariant "constituents" of the stress vector and an external force, e_{ij} are covariant components of the strain tensor, $\boldsymbol{U}$ is the displacement vector:

$$2e_{ij} = \boldsymbol{R}_i \partial_j \boldsymbol{U} + \boldsymbol{R}_j \partial_i \boldsymbol{U} + \partial_i \boldsymbol{U} \partial_j \boldsymbol{U}, \quad (\boldsymbol{R}_i = \partial_i \boldsymbol{R}) \tag{2}$$

$$\begin{aligned} E^{ijpq} &= \lambda g^{ij} g^{pq} + \mu (g^{ip} g^{jq} + g^{iq} g^{jp}), \quad (g^{ij} = \boldsymbol{R}^i \boldsymbol{R}^j), \\ E^{ijpqks} &= E_1 g^{ij} g^{pq} g^{ks} + E_2 g^{ij} g^{pk} g^{qs} + E_3 g^{ik} g^{pq} g^{js} + E_4 g^{ip} g^{jq} g^{ks}, \end{aligned} \tag{3}$$

where λ and μ are Lame's constants, E_1, E_2, E_3, E_4 are modules elasticity of the second order for isotropic elastic bodies, $\boldsymbol{R}_i$ and $\boldsymbol{R}^i$ are covariant and contravariant bases vectors of the surfaces $\widehat{S}(x^3 = const)$, which are "normally connected" with the basic vectors $\boldsymbol{r}_i$ and $\boldsymbol{r}^i$ of the midsurfaces $S(x^3 = 0)$ by the following relations:

$$\boldsymbol{R}(x^1, x^2, x^3) = \boldsymbol{r}(x^1, x^2) + x^3 \boldsymbol{n}(x^1, x^2), \quad (-h \le x^3 = x_3 \le h)$$

h is the thickness of a shell. Further

$$\begin{aligned} &\partial_\alpha \boldsymbol{R} = \boldsymbol{R}_\alpha = A_{\alpha.}^{.\beta} \boldsymbol{r}_\beta, \ \ \boldsymbol{R}^\alpha = A_{.\beta}^{\alpha.} \boldsymbol{r}^\beta, \ \ \boldsymbol{R}_3 = \boldsymbol{R}^3 = \boldsymbol{n}, \ \ \partial_\alpha \boldsymbol{r} = \boldsymbol{r}_\alpha, \ \ (\alpha, \beta = 1,2) \\ &\boldsymbol{r}_\alpha \boldsymbol{r}^\beta = \boldsymbol{R}_\alpha \boldsymbol{R}^\beta = \delta_\alpha^\beta, \ \ A_{\alpha.}^{.\beta} = a_\alpha^\beta - x^3 b_\alpha^\beta, \ \ \Lambda_{.\beta}^{\alpha.} = \vartheta^{-1} [a_\beta^\alpha + x_3 (b_\beta^\alpha - 2H a_\beta^\alpha)], \\ &a_{\alpha\beta} = \boldsymbol{r}_\alpha \boldsymbol{r}_\beta, \ \ b_{\alpha\beta} = -\boldsymbol{n}_\alpha \boldsymbol{r}_\beta, \ \ \vartheta = 1 - 2H x_3 + K x_3^2, \\ &2H = b_1^1 + b_2^2, \ \ K = b_1^1 b_2^2 - b_1^2 b_2^1. \end{aligned} \tag{4}$$

Note that sometimes under non-shallow shells the following approximation equalities

$$\boldsymbol{R}^\alpha \cong (a_\beta^\alpha + x_3 b_\beta^\alpha) \boldsymbol{r}^\beta, \ \ (W.\ Koiter,\ P.\ Naghdi,\ A.\ Lurie, ...) \tag{5}$$

are meant, which are the first approximations of the general case (4).

The relations (1-3) can be written as:

$$\nabla_\alpha(\vartheta\boldsymbol{\tau}^\alpha)+\partial_3(\vartheta\boldsymbol{\tau}^3)+\vartheta\boldsymbol{\Phi}=0, \tag{6}$$

$$\begin{gathered}\boldsymbol{\sigma}^i=\vartheta\boldsymbol{\tau}^i=\tfrac{\vartheta}{2}A^i_{i_1}[E^{i_1j_1p_1q_1}+\tfrac{1}{2}E^{i_1j_1p_1q_1k_1s_1}(A^k_{k_1}\boldsymbol{r}_{s_1}\partial_k\boldsymbol{U}+A^s_{s_1}\boldsymbol{r}_{k_1}\partial_s\boldsymbol{U}\\ +A^k_{k_1}A^s_{s_1}\partial_k\boldsymbol{U}\partial_s\boldsymbol{U})](A^p_{p_1}\boldsymbol{r}_{q_1}\partial_p\boldsymbol{U}+A^q_{q_1}\boldsymbol{r}_{q_1}\partial_q\boldsymbol{U}+A^p_{p_1}A^q_{q_1}\partial_p\boldsymbol{U}\partial_q\boldsymbol{U})(\boldsymbol{r}_{j_1}+A^j_{j_1}\partial_j\boldsymbol{U})\end{gathered} \tag{7}$$

where

$$E^{i_1j_1p_1q_1}=\lambda a^{i_1j_1}a^{p_1q_1}+\mu(a^{i_1p_1}a^{j_1q_1}+a^{i_1q_1}a^{j_1p_1}),\quad (a^{i_1j_1}=\boldsymbol{r}^{i_1}\boldsymbol{r}^{j_1}) \tag{8}$$

$$E^{i_1j_1p_1q_1k_1s_1}=a^{i_1j_1}(E_1a^{p_1q_1}a^{k_1s_1}+E_2a^{p_1k_1}a^{q_1s_1})+E_3a^{i_1k_1}a^{p_1q_1}a^{j_1s_1}+E_4a^{i_1p_1}a^{j_1q_1}a^{k_1s_1}.$$

Now we use I. Vekua's reduction method the essence of which consists, without going into details, in the following: since the system of Legendre polynomials $P_m\left(\frac{x_3}{h}\right)$ are complete in the interval $[-h,h]$, for equation (6) the equivalent infinite system of 2-D equations is obtained:

$$\nabla_\alpha\overset{(m)}{\boldsymbol{\sigma}}{}^\alpha-\frac{2m+1}{h}\left(\overset{(m-1)}{\boldsymbol{\sigma}}{}^3+\overset{(m-3)}{\boldsymbol{\sigma}}{}^3+...\right)+\overset{(m)}{\boldsymbol{F}}=0,\quad(\alpha=1,2;\ m=0,1,...) \tag{9}$$

where ∇_α are covariant derivatives on the midsurface $S(x_3=0)$.

Further

$$\left(\overset{(m)}{\boldsymbol{\sigma}}{}^i,\overset{(m)}{\boldsymbol{U}},\overset{(m)}{\boldsymbol{\Phi}}\right)=\frac{2m+1}{2h}\int\limits_{-h}^{h}\left(\vartheta\boldsymbol{\tau}^i,\boldsymbol{U},\vartheta\boldsymbol{\Phi}\right)P_m\left(\frac{x_3}{h}\right)dx_3, \tag{10}$$

where
$$\overset{(m)}{\boldsymbol{F}}=\overset{(m)}{\boldsymbol{\Phi}}+\frac{2m+1}{2h}\left(\overset{(+)}{\vartheta}\overset{(+)}{\boldsymbol{\sigma}}{}_3-(-1)^m\overset{(-)}{\vartheta}\overset{(-)}{\boldsymbol{\sigma}}{}_3\right),$$

$$\overset{(\pm)}{\vartheta}=1\mp2hH+Kh^2,\quad\overset{(\pm)}{\boldsymbol{\sigma}}{}^3=\boldsymbol{\sigma}^3(x^1,x^2,\pm h).$$

For $\overset{(m)}{\boldsymbol{\sigma}}{}^i$ we have

$$\begin{aligned}\overset{(m)}{\boldsymbol{\sigma}}{}^i=&\frac{1}{2}E^{i_1j_1p_1q_1}\sum_{m_1=0}^{\infty}\Bigg[\underset{(m_1)}{\overset{(m)}{A}}{}^{i\,p}_{i_1p_1}\Big(\boldsymbol{r}_{q_1}D_p\overset{(m_1)}{\boldsymbol{U}}\Big)\boldsymbol{r}_{j_1}+\sum_{m_2=0}^{\infty}\Big(\underset{(m_1,m_2)}{\overset{(m)}{A}}{}^{i\,pq}_{i_1p_1q_1}\Big(D_p\overset{(m_1)}{\boldsymbol{U}}D_q\overset{(m_2)}{\boldsymbol{U}}\Big)\boldsymbol{r}_{j_1}\\&+\cdots+\sum_{m_3=0}\underset{(m_1,m_2,m_3)}{\overset{(m)}{A}}{}^{i\,j\,p\,q}_{i_1j_1p_1q_1}\Big(D_p\overset{(m_1)}{\boldsymbol{U}}D_q\overset{(m_2)}{\boldsymbol{U}}\Big)D_j\overset{(m_3)}{\boldsymbol{U}}\Big)\Bigg]+\cdots+\\&+\frac{1}{4}\sum_{m_1,\dots m_5}E^{i_1j_1p_1q_1k_1s_1}\underset{(m_1,m_2,m_3)}{\overset{(m)}{A}}{}^{i\,j\,p\,q\,ks}_{i_1j_1p_1q_1k_1s_1}\Big(D_p\overset{(m_1)}{\boldsymbol{U}}D_q\overset{(m_2)}{\boldsymbol{U}}\Big)\Big(D_k\overset{(m_3)}{\boldsymbol{U}}D_s\overset{(m_4)}{\boldsymbol{U}}\Big)D_j\overset{(m_5)}{\boldsymbol{U}},\end{aligned} \tag{11}$$

where
$$D_i\overset{(m)}{\boldsymbol{U}}=\delta^\beta_i\partial_\beta\overset{(m)}{\boldsymbol{U}}+\delta^3_i\overset{(m)}{\boldsymbol{U}}{}';\quad\overset{(m)}{\boldsymbol{U}}{}'=\frac{2m+1}{h}\left(\overset{(m+1)}{\boldsymbol{U}}+\overset{(m+3)}{\boldsymbol{U}}+...\right),$$

$$\underset{(m_1)}{\overset{(m)}{A}}{}^{ij}_{i_1j_1}=\frac{2m+1}{2h}\int\limits_{-h}^{h}\vartheta A^i_{i_1}A^j_{j_1}P_{m_1}P_mdx_3,\quad\cdots. \tag{12}$$

Now me have the following integrals:

a) For the shallow shells

$$\int\limits_{-h}^{h} P_s P_m dx_3, \quad \int\limits_{-h}^{h} P_{s_1} P_{s_2} P_m dx_3, \quad \int\limits_{-h}^{h} P_{s_1} P_{s_2} P_{s_3} P_m dx_3.$$

It should be noted that those integrals can be calculated by means of Adams formulas:

$$P_m(x)P_n(x) = \sum_{r=0}^{\min(m,n)} \frac{A_{m-r}A_r A_{n-r}}{A_{m+n-r}} \frac{2(m+n)-4r+1}{2(m+n)-2r+1} P_{m+n-2r}(x),$$

$$A_m = \frac{1\cdot 3\cdots(2m-1)}{m!}.$$

b) For non-shallow shells (Koiter-Naghdi, Lurie,...) we have the following integrals:

$$\int\limits_{-h}^{h} x_3^k P_{s_1}\cdots P_{s_n} P_m dx_3, \quad (k=0,1,...,6;\ n=1,2,...,5).$$

c) For non-shallow shells (I. Vekua,...) we have integrals of the types

$$\int\limits_{-h}^{h} \frac{x_3^k P_{s_1}\cdots P_{s_n} P_m}{(1-2Hx_3+Kx_3^2)^n} dx_3, \quad (k=0,1,...,6;\ n=1,2,...,5)$$

which are calculated by means of Adams and Niemann formulas

$$\int\limits_{-1}^{1} \frac{P_m(y)dt}{x-y} = 2Q_m(x), \quad (|x|>1),$$

where $Q_m(x)$ is the Legender function of the second kind.

For example we have the integrals of type (12) which can be calculated

$$\overset{(m)}{\underset{(m_1)}{A}}{}^{\alpha\beta}_{\alpha_1\beta_1} = \frac{L^\alpha_{\alpha_1} L^\beta_{\beta_1}}{K}\delta^m_{m_1} + \frac{2m+1}{2\sqrt{E}h}\left[B^\alpha_{\alpha_1}(hy)B^\beta_{\beta_1}(hy)\begin{pmatrix} P_{m_1}(y)Q_m(y),\ m_1\le m \\ Q_{m_1}(y)P_m(y),\ m_1\le m\end{pmatrix}\right]_{y_1}^{y_2},$$

where $B^\alpha_\beta(x) = a^\alpha_\beta + xL^\alpha_\beta$, $L^\alpha_\beta = b^\alpha_\beta - 2Ha^\alpha_\beta$, $E = H^2 - K$, $y_{1,2} = [(H \mp \sqrt{E})h]^{-1}$.

Let $\rho = \max(b^\alpha_\beta, b^{\alpha\beta}, b_{\alpha\beta}) \ \Rightarrow h < \rho \Rightarrow \frac{h}{\rho} = \varepsilon < 1 \Rightarrow |\varepsilon b^\alpha_\beta \rho| \le q < 1$, where ε is a small parameter, h is the semithickness of a shell.

Now, following Signorini [2], we assume the validity of the expansions for approximations of order N:

$$\left(\overset{(m)}{\boldsymbol{\sigma}}{}^i, \overset{(m)}{\boldsymbol{U}}, \overset{(m)}{\boldsymbol{F}}\right) = \sum_{n=1}^{\infty} \left(\overset{(m,n)}{\boldsymbol{\sigma}}{}^i, \overset{(m,n)}{\boldsymbol{U}}, \overset{(m,n)}{\boldsymbol{F}}\right)\varepsilon^n, \quad (m=0,1,...,N).$$

Substituting the above expansions into relation (9) we obtain the following 2-D finite system of equilibrium equations to components of displacement vector of the isometric coordinates:

$$\begin{aligned}
&4\mu\partial_{\bar z}\Big(\Lambda^{-1}\partial_z \overset{(m,n)}{U}_+\Big)+2(\lambda+\mu)\partial_{\bar z}\overset{(m,n)}{\Theta}+\frac{2\lambda}{h}\partial_{\bar z}\overset{(m,n)}{U_3'}\\
&-\frac{2m+1}{h}\mu\Big[2\partial_{\bar z}\Big(\overset{(m-1,n)}{U}_3+\overset{(m-3,n)}{U}_3+\cdots\Big)+\overset{(m-1,n)}{U'_+}+\overset{(m-3,n)}{U'_+}+\cdots\Big]+\overset{(m,n)}{F}_+=0,\\
&\mu\Big(\nabla^2\overset{(m,n)}{U}_3+\overset{(m,n)}{\Theta'}\Big)-\frac{2m+1}{h}\Big[\lambda\Big(\overset{(m-1,n)}{\Theta}+\overset{(m-3,n)}{\Theta}+\cdots\Big)\\
&+(\lambda+2\mu)\Big(\overset{(m-1,n)}{U'_3}+\overset{(m-3,n)}{U'_3}+\cdots\Big)\Big]+\overset{(m,n)}{F}_3=0,
\end{aligned}\tag{13}$$

where $\qquad ds^2=\Lambda(z,\bar z)dzd\bar z,\quad \Big(z=x^1+ix^2,\ \ 2\partial_{\bar z}=\partial_1-i\partial_2,\ \ \nabla^2=\dfrac{4}{\Lambda}\dfrac{\partial^2}{\partial z\partial\bar z}\Big),$

$$U_+=U_1+iU_2,\ \ \Theta=\Lambda^{-1}\Big(\partial_z U_+ +\partial_{\bar z}\bar U_+\Big).$$

Note that system (13) can be rewritten as

$$\mu\Delta\overset{(m)}{U}_+ +2(\lambda+\mu)\partial_{\bar z}\overset{(m)}{\Theta}+\overset{(m)}{L}_+\Big(\overset{(0)}{U}_i,...,\overset{(N)}{U}_i\Big)=0,\ \ (L_+=L_1+iL_2),\tag{14}$$

$$\mu\Delta\overset{(m)}{U}_3+\overset{(m)}{L}_3\Big(\overset{(0)}{U}_i,...,\overset{(N)}{U}_i\Big)=0,\ \ (i=1,2,3),\ \ m=0,1,...,N,\tag{15}$$

where the main part of equation (14) is the operator of the plane theory of elasticity and the main part of equation (15) is the Laplace $\Delta=4\dfrac{\partial^2}{\partial z\partial\bar z}$ operator.

R E F E R E N C E S

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Author's address:

T. Meunargia
I. Vekua Institute of Applied Mathematics of
Iv. Javakhishvili Tbilisi State University
2, University St., Tbilisi 0186
Georgia
E-mail: tengiz.meunargia@viam.sci.tsu.ge