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ASYMPTOTIC BEHAVIOR OF SOLUTION AND SEMI-DISCRETE DIFFERENCE SCHEME FOR ONE NONLINEAR INTEGRO-DIFFERENTIAL EQUATION WITH SOURCE TERM

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Abstract. One nonlinear partial integro-differential equation with source term is considered. The model arises at describing penetration of a magnetic field into a substance and is based on the Maxwell system. Large time behavior of solution of the initial-boundary value problem as well as semi-discrete finite scheme are studied. More wide class of nonlinearity is considered than one has been already investigated.

Keywords and phrases: Nonlinear integro-differential equation, asymptotic behavior, semi-discrete finite difference scheme.

AMS subject classification: 45K05, 65M06.

Mathematical simulation of many applied problems leads to nonlinear integro-differential equations (see, for example, [1]-[9] and reference therein). One kind of nonlinear partial integro-differential model were introduced in [10] after reduction of nonlinear Maxwell's differential system [11]. In [12] some generalizations of such type integro-differential models are given. One-dimensional simple analog with source term called by averaged integro-differential model has the following form

$$\frac{\partial U}{\partial t} - a\left(\int_0^t \int_0^1 \left(\frac{\partial U}{\partial x}\right)^2 dx d\tau\right) \frac{\partial^2 U}{\partial x^2} + f(U) = 0, \tag{1}$$

where $a = a(S)$ and $f = f(U)$ are given functions of their arguments.

Many works are dedicated to the investigation and numerical resolution of the integro-differential models described in [10] and [12]. Especially, in [10], [12]-[17] solvability and uniqueness of the initial-boundary value problems for these type equations are studied. Asymptotic behavior of solutions as $t \to \infty$ is investigated in many works also (see, for example, [18]-[23] and references there in). Numerical resolution by finite difference schemes, Galerkin's and finite element methods are given in works [18]-[24] and in a number of other works as well.

The aim of this note is to study asymptotic behavior of solution as $t \to \infty$ and investigate convergence of corresponding semi-discrete scheme for the equation (1).

In the $[0, 1] \times [0, \infty)$ let us consider following initial-boundary value problem:

$$\begin{gathered} U(0,t) = U(1,t) = 0, \\ U(x,0) = U_0(x), \end{gathered} \tag{2}$$

where $U_0 = U_0(x)$ is given function.

The following statement is true.

Theorem 1. *If $a = a(S) \geq a_0 = Const > 0$, $a^{'}(S) \geq 0$, $a^{''}(S) \leq 0$, $f(U) = |U|^{q-2}U$, $q \geq 2$ and $U_0 \in H_0^1(0,1)$, then problem (1),(2) has not more than one solution and the following asymptotic property takes place*

$$\left\| \frac{\partial U}{\partial x} \right\| \leq C \exp\left(-\frac{a_0 t}{2} \right).$$

Here $\|\cdot\|$ is norm of the space $L_2(0,1)$, $H_0^1(0,1)$ is the usual Sobolev space and C denotes positive constant independent of t.

On [0,1] let us introduce a net with mesh points denoted by $x_i = ih$, $i = 0, 1, \ldots, M$, with $h = 1/M$. The boundaries are specified by $i = 0$ and $i = M$. The semi-discrete approximation at (x_i, t) is designed by $u_i = u_i(t)$. The exact solution to the problem at (x_i, t) is denoted by $U_i = U_i(t)$. At points $i = 1, 2, \ldots, M-1$, the integro-differential equation will be replaced by approximation of the space derivatives by a forward and backward differences. We will use the following known notations:

$$u_{x,i}(t) = \frac{u_{i+1}(t) - u_i(t)}{h}, \quad u_{\bar{x},i}(t) = \frac{u_i(t) - u_{i-1}(t)}{h}.$$

Let us correspond to problem (1),(2) the following semi-discrete scheme:

$$\frac{du_i}{dt} = \left\{ a\left(h \sum_{i=1}^{M} \int_0^t (u_{\bar{x},i})^2 \, d\tau \right) u_{\bar{x},i} \right\}_x + |u_i|^{q-2} u_i, \quad i = 1, 2, \ldots, M-1, \tag{3}$$

$$u_0(t) = u_M(t) = 0, \tag{4}$$

$$u_i(0) = U_{0,i}, \quad i = 0, 1, \ldots, M. \tag{5}$$

So, we obtained Cauchy problem (3)-(5) for nonlinear system of ordinary integro-differential equations.

Introduce usual discrete inner product and norm:

$$(u, v)_h = h \sum_{i=1}^{M-1} u_i v_i, \quad \|u\|_h = (u, u)_h^{1/2}.$$

The following statement takes place.

Theorem 2. *If $a = a(S) \geq a_0 = Const > 0$, $a^{'}(S) \geq 0$, $a^{''}(S) \leq 0$, $f(U) = |U|^{q-2}U$, $q \geq 2$ and problem (1),(2) has a sufficiently smooth solution $U = U(x,t)$, then the solution $u = u(t) = (u_1(t), u_2(t), \ldots, u_{M-1}(t))$ of problem (3)-(5) tends to $U = U(t) = (U_1(t), U_2(t), \ldots, U_{M-1}(t))$ as $h \to 0$ and the following estimate is true*

$$\|u(t) - U(t)\|_h \leq Ch.$$

Here C denotes positive constant independent of h.

Note that for solving the finite difference scheme corresponding to (3)-(5) an algorithm analogical to [20] is used. So, it is necessary to use Newton iterative process [25].

According to this method the great numbers of numerical experiments are carried out. These experiments agree with the theoretical results given in the Theorems 1 and 2.

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