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ON THE ABSOLUTE CONVERGENCE OF MULTIPLE FOURIER–HAAR SERIES

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Abstract. The analogues of Szasz's theorem for s-dimensional Fourier–Haar series is given. It is shown that this theorem when $\alpha = 1$ is best possible for general complete orthonormal systems.

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Let R^s, $s \geq 1$, be the Euclidean space and let N^s be the set of lattice points in R^s. We denote by $X = (x_1, \dots, x_s)$, $Y = (y_1, \dots, y_s)$ the points of the space R^s and by $M = (m_1, \dots, m_s)$, $T = (t_1, \dots, t_s)$ the points of the set N^s. If B is an arbitrary nonempty subset of the set $\{1, \dots, s\} = \overline{S}$, then we denote by X_B the points $(x'_1, \dots, x'_s)$ where $x'_i = x_i$ for $i \in B$ and $x'_i = 0$ for $i \in \overline{S} \setminus B$. We will use the following notation: Π_s is the set of all nonempty subsets of the set $\overline{S}$; $|B|$ is the number of elements of the set B; $E = (1, \dots, 1)$; $I^s = [0, 1]^s$; $C(I^s)$ is the space of continuous functions; $\|f\|_{C(I^s)}$ is the norm of f in the space $C(I^s)$.

Let $f \in C(I^s)$. We set

$$\Delta_{H_{\{i\}}} f(x) = f(X + H_{\{i\}}) - f(X)$$

and define $\Delta_{H_B} f(x)$ to be repeated applications of the operation $H_{\{i\}}$ when i runs over the set $B \subset \overline{S}$.

Let $r_n(x)$ and $\chi_n(x)$ be, respectively, the Rademacher and Haar functions (see [1], Ch. 1, Sections 6, 7).

Now let us define the Rademacher multiple functions as

$$r_N(X) = \prod_{i=1}^{s} r_{n_i}(x_i)$$

and the Haar multiple functions as

$$\chi_N(X) = \prod_{i=1}^{s} \chi_{n_i}(x_i).$$

Definition 1. Denote by $H_\alpha^{(s)}$ the set of all functions continuous on I^s, for which

$$\sup_{|t_{i_j}| \leq |h_{i_j}|} \|\Delta_{T_B} f(x)\|_{C(I^s)} = O\Big(\prod_{i_j \in B} |h_{i_j}|^\alpha\Big)$$

for any $B = \{i_1, \dots, i_{|B|}\} \in \Pi_s$.

Definition 2. Denote by $\operatorname{Lip}\alpha$, $\alpha \in (0,1]$, the set of all functions continuous on I^s, for which

$$\sup_{\substack{|l_i|\le|h_i|\\1\le i\le s}} \|f(X+T)-f(X)\|_{C(I^s)} = O(|H|^\alpha),$$

where $|H| = (h_1^2+\cdots+h_s^2)^{\frac{1}{2}}$.

The following lemma holds true

Lemma. *If $f(X)\in \operatorname{Lip}\alpha$, $\alpha\in(0,1]$, then $f(X)\in H^{(s)}_{\frac{\alpha}{s}}$.*

The following theorem is a multidimensional analogue of the corresponding theorem in [2] (see Ch. 7, Section 4).

Theorem 1. *Let $(f_N(X))$ be a sequence of functions such that $\|f_N\|_{C(I^s)} < C$ (C does not depend on N). Then if*

$$\sum_{N=E}^{\infty} a_N^2 < +\infty,$$

the function

$$f(X,T) = \sum_{N=E}^{\infty} a_N f_N(X) r_n(X)$$

belongs to $L_p(I^s)$ for almost every $T\in I^s$.

The following theorem for $s=1$ was proved by B.I. Golubov [3].

Theorem 2. *Let $f\in H^{(s)}_{\frac{\beta}{s}}$ for every $\beta\in(0,\alpha)$, $\alpha\in(0,1]$. Then for any $\varepsilon>0$*

$$\sum_{N=E}^{\infty} |\widehat{\chi}_N(f)|^{\frac{2s}{s+2\alpha}+\varepsilon} < +\infty.$$

Theorem 3. *Let $p>2$ and let $(\varphi_N(X))$ be any complete orthonormal system (ONS) in $L_2(I^s)$. Then there exists a function $f(X)$ such that $f'_{x_i}\in L_p(I^s)$, $i=1,\dots,s$, and*

$$\sum_{N=E}^{\infty} |\widehat{\varphi}_N(f)|^{\frac{2s}{s+2}} = +\infty.$$

This theorem is the s-dimensional analogue of Szasz's [4] theorem for the Haar series.

The following theorem shows that Theorem 2 is best possible when $\alpha=1$.

Theorem 4. *Let $(\varphi_N(X))$ be any complete ONS in $L_2(I^s)$. Then there exists a function $f(X)\in\operatorname{Lip}\alpha$ for every $\alpha\in(0,1)$ such that*

$$\sum_{N=E}^{\infty} |\widehat{\varphi}_N(f)|^{\frac{2s}{s+2}} = +\infty.$$

In the special case $\varphi_N(X)=\chi_n(X)$ Theorem 4 was proved by B.I. Golubov [3].

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