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ESTIMATION OF FOURIER COEFFICIENTS OF CONTINUOUS FUNCTIONS WITH RESPECT TO GENERAL ORTHONORMAL SYSTEMS

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Abstract. The conditions are found which should be satisfied by the functions of ONS $(\varphi_n(x))$ so that Fourier coefficients of continuous functions or functions with bounded variation have the same estimates with respect to this system as they have with respect to the trigonometric system.

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As is known, $\omega(\delta, f) = \sup\limits_{|x-y|\le\delta} \|f(x)-f(y)\|_C$ $(x, y \in [0,1])$ is the continuity modulus of function $f(x) \in C(0,1)$ with respect to the norm $C(0,1)$ and

$$\omega^{(2)}(\delta, f) = \sup_{|x-y|\le\delta} \left\| f(x) - 2f\Big(\frac{x+y}{2}\Big) + f(y) \right\|_C$$

is the smoothness modulus of function $f(x) \in C(0,1)$.

If $f(x) \in L_2(0,1)$, then

$$\omega_2(\delta, f) = \sup_{|h|\le\delta} \left(\int_0^{1-h} |f(x) - f(x+h)|^2 \, dx \right)^{\frac{1}{2}}$$

is the integral module of function $f(x)$.

Let $(\varphi_n(x))$ be an orthonormal on $[0,1]$ system of functions (ONS) and let $\widehat{\varphi}(f) = \int_0^1 f(x)\varphi_n(x)\,dx$ $(n = 1, 2, \dots)$ be the Fourier coefficients of functions $f(x) \in L(0,1)$ with respect to the system $(\varphi_n(x))$.

As is known (see [1], p. 80–81, and [2], p. 79) for periodic functions $f(x)$ with the period 2π from $C(0, 2\pi)$ the following estimates are valid:

$$\text{a)} \quad a_n(f) = O(1)\omega\left(\frac{1}{n}, f\right) \quad \text{and} \quad b_n(f) = O(1)\omega\left(\frac{1}{n}, f\right), \tag{1}$$

where

$$a_n(f) = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx \quad \text{and} \quad b_n(f) = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx;$$

$$\text{b)} \quad a_n(f) = O(1)\omega^{(2)}\left(\frac{1}{n}, f\right) \quad \text{and} \quad b_n(f) = O(1)\omega^{(2)}\left(\frac{1}{n}, f\right). \tag{2}$$

For the Haar system (see [3]) the estimate of type (1) holds true, but in the case of functions $f_0(x) = x$ it is seen that estimate (2) does not hold true.

As to the estimate of Fourier coefficients for general ONS with respect to some classes of functions, the following theorems are true.

Theorem 1. *Let* $(\varphi_n(x))$ *be ONS on* $[0,1]$ *and* $\int_0^1 \varphi_n(x)\,dx = 0$, $n = k_0, k_0+1, \ldots$, *where* k_0 *is some number. For any function* $f(x) \in C(0,1)$ *the relation*

$$\widehat{\varphi}_n(f) = O(1)\omega\left(\frac{1}{n}, f\right)$$

is valid if and only if

$$\mathrm{V}_n = n\int_0^1 \left|\int_0^x \varphi_n(t)\,dt\right| dx = O(1).$$

We will note here that for the trigonometric system $(T_n(x))$ and the Haar system $(\chi_n(x))$ it is easily verified that

$$\mathrm{V}_n = O(1).$$

Theorem 2. *Let* $(\varphi_n(x))$ *be ONS on* $[0,1]$, $\int_0^1 \varphi_n(x)\,dx = O\big(\frac{1}{n}\big)$, $n = 1, 2, \ldots$, *and* $\sup\limits_{x\in[0,1]} |\varphi_n(x)| < M$. *For any function* $f(x)$ *of finite variation the condition*

$$\widehat{\varphi}_n(f) = O\left(\frac{1}{n}\right)$$

holds true if and only if

$$B_n = \max_{x\in[0,1]} \left|\int_0^x \varphi_n(t)\,dt\right| = O(1).$$

It should be noted that for the trigonometric system the condition $B_n = O(1)$ is true, however, for the Haar system $B_n \neq O(1)$.

Below Fourier coefficients of some classes of functions with respect to general ONS will be estimated by means of $\omega^{(2)}(\delta, f)$ and $\omega_2(\delta, f)$.

From [4] it can be concluded that for general ONS, estimates (1) and (2) do not hold, even for those functions which have continuous derivatives.

Below the results are given which show that in some cases when the functions of ONS $(\varphi_n(x))$ satisfy certain conditions, estimates (2) hold.

Introduce the notation:

$$\Phi_n(x) = \int_0^x \varphi_n(t)\,dt \quad \text{and} \quad H_n = n\sum_{k=1}^{n-1} \left|\int_0^{k/n} \Phi_n(x)\,dx\right|.$$

Theorem 3. *Let* $\int_0^1 \varphi_n(x)\,dx = 0$, $\int_0^1 \Phi_n(x)\,dx = 0$, $n = 1, 2, \ldots$, *and* $H_n = O(1)$. *Then for any function* $f'(x) \in L_2(0,1)$ *the following estimate holds:*

$$\widehat{\varphi}_n(f) = O(1)\left(\omega^{(2)}\Big(\frac{1}{n}, f\Big) + \frac{1}{\sqrt{n}}\,\omega_2\Big(\frac{1}{n}, f'\Big)\right).$$

Theorem 4. *If ONS* $(\varphi_n(x))$ *satisfied the conditions:*

a) $\int_0^1 \varphi_n(x)\,dx = 0$ $(n = 1, 2, \dots)$*;*

b) *for any* $f(x) \in C(0,1)$, $\widehat{\varphi}_n(f) = O(1)\omega^{(2)}\left(\frac{1}{n}, f\right)$,

then

$$\int_0^1 \Phi_n(x)\,dx = 0 \quad (n = 1, 2, \dots).$$

Remark 1. For the trigonometric system $H_n = O(1)$.

Remark 2. For the Haar system the condition $H_n = O(1)$ does not hold.

R E F E R E N C E S

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