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CONVERGENCE OF DOUBLE TRIGONOMETRIC SERIES OBTAINED BY TERMWISE INTEGRATION

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Abstract. It is shown that for every 2π periodic in each variable function f of two variables, summable on the square $[0, 2\pi]^2$, termwise integrating its double trigonometric Fourier series on the rectangle $[0, x] \times [0, y]$ gives a uniformly converging on $[0, 2\pi]^2$ to the integral $\int_0^x \int_0^y f(t, \tau)\, dt\, d\tau$ series. A series sum $\sum\limits_{m,n=1}^{\infty} b_{mn}/mn$ is found, where b_{mn} is the Fourier coefficient at the product $\sin mx \sin ny$.

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1. Among many problems considered by Bernard Riemann there was the problem of representation of a function by a trigonometric series (1854). To solve this problem, Riemann considered the series with bounded coefficients

$$c_0 + \sum_{|n|\geq 1} c_n e^{inx}, \tag{1}$$

and by twice integrating it formally he obtained an everywhere continuous function

$$F(x) = c_0 \frac{x^2}{2} + \sum_{|n|\geq 1} \frac{1}{n^2} c_n e^{inx}.$$

Riemann introduced the second symmetric derivative (later called a derivative in the Schwarz sense) which is written in the form

$$F^{(\prime\prime)}(x) = \lim_{h\to 0} \frac{F(x+2h) + F(x-2h) - 2F(x)}{4h^2},$$

for the function F, and in the form

$$F^{(\prime\prime)}(x) = \lim_{h\to 0} \left[c_0 + \sum_{|n|\geq 1} c_n e^{inx} \left(\frac{\sin nh}{nh} \right)^2 \right]$$

while for series (1). $F^{(\prime\prime)}(x)$ is called the sum of series (1) in the Riemann sense.

2. Riemann's idea about a formally integrated series was used by Lebesgue, who performed the operation of single formal integration of series (1) and obtained the series

$$c_0 x - i \sum_{|n|\geq 1} \frac{1}{n} c_n e^{inx}. \tag{2}$$

If series (2) converges to the function $\ell(x)$ in the neighborhood of some point x_0 and $\ell(x)$ has, at the point x_0, the symmetric derivative

$$\ell^{(\prime)}(x_0) = \lim_{h\to 0} \frac{1}{2h}\,[\ell(x_0+h) - \ell(x_0-h)],$$

then $\ell^{(\prime)}(x_0)$ is called the sum of series (1) in the Lebesgue sense at the point x_0, which according to series (1) is written in the following form

$$\ell^{(\prime)}(x_0) = \lim_{h\to 0}\left[c_0 + \sum_{|n|\ge 1} c_n e^{inx_0}\frac{\sin nh}{nh}\right].$$

Despite the well-known fact that there exists a summable function,the Fourier series of which diverges everywhere (Kolmogorov's example), the sum of the Fourier series $S[f]$ in the Riemann and Lebesgue sense will be equal to the values of f for every function f almost at all points.This fact was established by Lebesgue by means of the following theorem proved by him in 1902.

Theorem L. *If the Fourier series of a 2π periodic and summable function f on $[0,2\pi]$ are, respectively,*

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n \sin nx) \quad \text{and} \quad f\sim c_0 + \sum_{|n|\ge 1}^{\infty} c_n e^{inx},$$

then the following equalities are fulfilled uniformly on $[0,2\pi]$, respectively,

$$\int_0^x f(t)dt = \begin{cases} \dfrac{a_0}{2}\displaystyle\int_0^x dt + \sum_{n=1}^{\infty}\int_0^x (a_n\cos nt + b_n\sin nt)\,dt, \\ \displaystyle\sum_{n=1}^{\infty}\frac{b_n}{n} + \frac{a_0}{2}x + \sum_{n=1}^{\infty}\frac{1}{n}(a_n\sin nx - b_n\cos nx) \end{cases}$$

and

$$\int_0^x f(t)dt = \begin{cases} c_0\displaystyle\int_0^x dt + \sum_{|n|\ge 1}^{\infty}\int_0^x c_n e^{int}dt, \\ i\displaystyle\sum_{|n|\ge 1}\frac{c_n}{n} + c_0 x - i\sum_{|n|\ge 1}^{\infty}\frac{1}{n}\,c_n e^{ix}. \end{cases}$$

Moreover, the following equalities are fulfilled, too:

$$\sum_{|n|\ge 1}\frac{c_n}{n} = -i\sum_{n=1}^{\infty}\frac{b_n}{n}, \quad \sum_{n=1}^{\infty}\frac{b_n}{n} = \frac{1}{2\pi}\int_0^{2\pi}(\pi - x)f(x)\,dx.$$

3. Our objectives are: 1) to investigate the existence of an analogous theorem to the Lebesgue theorem L for double Fourier series; 2) to consider the convergence in the Lebesgue sense of double Fourier series, keeping in mind the fact that there exists a

2π periodic in each variable and everywhere continuous function of two variables, the Fourier series of which diverges everywhere [1]. Hence the following theorems are valid.

Theorem 1. *For the exponential series of a 2π periodic in each variable and summable function f on $[0,2\pi]^2$*

$$f \sim c_{00} + \sum_{|m|\ge 1} c_{m0} e^{imx} + \sum_{|n|\ge 1} c_{0n} e^{iny} + \sum_{|m|\ge 1,\, |n|\ge 1} c_{mn} e^{i(mx+ny)}, \tag{3}$$

the equality

$$\int_0^x \int_0^y f(t,\tau)\, dt\, d\tau = c_{00}xy + iy \sum_{|m|\ge 1} \frac{1}{m} c_{m0}(1-e^{imx}) + ix \sum_{|n|\ge 1} \frac{1}{n} c_{0n}(1-e^{iny}) - \sum_{|m|\ge 1,\, |n|\ge 1} \frac{1}{mn} c_{mn}(1-e^{imx})(1-e^{iny})$$

is fulfilled uniformly on $[0,2\pi]^2$.

Corollary 1. The equality

$$\sum_{|m|\ge 1,\, |n|\ge 1} \frac{c_{mn}}{mn} = -\sum_{m,n=1}^{\infty} \frac{b_{mn}}{mn} \tag{4}$$

is valid, where b_{mn} is the Fourier coefficient at $\sin mx \sin ny$ from the relation

$$f \sim \frac{1}{4} a_{00} + \frac{1}{2} \sum_{m=1}^{\infty} (a_{m0} \cos mx + d_{m0} \sin mx) + \frac{1}{2} \sum_{n=1}^{\infty} (a_{0n} \cos ny + c_{0n} \sin ny) + \sum_{m,n=1}^{\infty} (a_{mn} \cos mx \cos ny + b_{mn} \sin mx \sin ny + c_{mn} \cos mx \sin ny + d_{mn} \sin mx \cos ny). \tag{5}$$

4. The left-hand side of equality (4) will be known if we know the right-hand side of the same equality.

The study of this issue showed that in order to find the right-hand side of equality (4) it is necessary to prove an analogue of Theorem 1 for series (5). In this context, the following statement is true.

Theorem 2. *If f is a 2π periodic in each variable and summable function on $[0,2\pi]^2$, then for series* (5) *the equality*

$$\int_0^x \int_0^y f(t,\tau)\, dt\, d\tau = \frac{1}{4} a_{00}xy + \frac{1}{2} y \sum_{m=1}^{\infty} \int_0^x (a_{m0} \cos mt + d_{m0} \sin mt)\, dt + \frac{1}{2} x \sum_{n=1}^{\infty} \int_0^y (a_{0n} \cos n\tau + c_{0n} \cos n\tau)\, d\tau + \sum_{m,n=1}^{\infty} \int_0^x \int_0^y [a_{mn} \cos mt \cos n\tau + b_{mn} \sin mt \sin n\tau + c_{mn} \cos mt \sin n\tau + d_{mn} \sin mt \cos n\tau]\, dt\, d\tau$$

is fulfilled uniformly on $[0, 2\pi]^2$.

Corollary 2. The equality

$$\sum_{m,n=1}^{\infty} \frac{b_{mn}}{mn} = -\frac{1}{4}\left(A_{00} + a_{00}\pi^2\right) - \frac{1}{2}\sum_{m=1}^{\infty}\frac{1}{m}\,\beta_m + \frac{\pi}{2}\sum_{m=1}^{\infty}\frac{1}{m}\,d_{m0}$$

$$-\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}\,\delta_n + \frac{\pi}{2}\sum_{n=1}^{\infty}\frac{1}{n}\,c_{0n}, \tag{6}$$

is fulfilled, where

$$\beta_m = \frac{1}{\pi^2}\int_0^{2\pi}\int_0^{2\pi} y f(x,y)\sin mx\,dx\,dy, \quad \delta_n = \frac{1}{\pi^2}\int_0^{2\pi}\int_0^{2\pi} x f(x,y)\sin ny\,dx\,dy,$$

$$F(x,y) = \int_0^x\int_0^y f(t,\tau)\,dt\,d\tau - y\int_0^x\left(\frac{1}{2\pi}\int_0^{2\pi} f(t,y)dy\right)dt - x\int_0^y\left(\frac{1}{2\pi}\int_0^{2\pi} f(x,\tau)dx\right)d\tau,$$

$$A_{00} = \frac{1}{\pi^2}\int_0^{2\pi}\int_0^{2\pi} F(x,y)\,dx\,dy.$$

Corollary 3. The series

$$\sum_{m,n=1}^{\infty}\left(\frac{c_{mn}}{mn}\sin mx - \frac{b_{mn}}{mn}\cos mx\right) \quad \text{and} \quad \sum_{m,n=1}^{\infty}\left(\frac{d_{mn}}{mn}\sin ny - \frac{b_{mn}}{mn}\cos ny\right)$$

are convergent on the segments $0 \le x \le 2\pi$ and $0 \le y \le 2\pi$.

5. The sum of Fourier series (3) in the Lebesgue sense can be characterized for various classes of functions. We have the following theorem as an example.

Theorem 3. *If a* 2π *periodic in each variable and summable on* $[0, 2\pi]^2$ *function* f *has a continuity point* (x_0, y_0)*, then the sum of (3) in the Lebesgue sense is equal to* $f(x_0, y_0)$.

Corollary 4. For Fefferman's function (see [1]), series (3) converges in the Lebesgue sense to $f(x,y)$ at all points (x,y).

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