Reports of Enlarged Session of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 28, 2014

THE ABEL SUMMABILITY OF CONJUGATE LAPLACE SERIES

Caramuta P., Cialdea A., Silverio F.

Abstract. In the present paper we describe the concept of conjugate Laplace series and present some results concerning its Abel summability.

Keywords and phrases: Laplace series, conjugate series, Abel summability, differential forms.

AMS subject classification: 33C55, 40G10, 58A10, 42A50.

1. Introduction. The classical theory of conjugate Fourier series is well known (see, e.g. [1]). It is possible to extend the concept of conjugate series in higher dimensions in different ways. Muckenhoupt and Stein gave a concept of conjugate ultraspherical expansion in [2], which later was generalized to Jacobi series by Li [3]. Cialdea introduced a different concept of conjugate Laplace series in [4]. It hinges on the notion of conjugate differential forms, which is an extension of the classical definition of conjugate harmonic functions. In the case $n = 3$, if

$$\sum_{h=0}^{\infty} \sum_{k=0}^{2h} a_{hk} Y_{hk}(\phi, \theta)$$

is a spherical expansion, its conjugate series, according to [4], is

$$\sum_{h=1}^{\infty} \sum_{k=0}^{2h} \frac{a_{hk}}{h+1} \left[\frac{1}{\sin\phi} \frac{\partial Y_{hk}}{\partial \theta} d\phi - \sin\phi \frac{\partial Y_{hk}}{\partial \phi} d\theta \right]. \tag{1}$$

We remark that (1) is not a series of scalar functions, but a series of differential forms of degree one on the unit sphere. In general n-dimensional case, it is a series of differential forms of degree $n-2$ on $\Sigma = \{x \in \mathbb{R}^n : |x| = 1\}$. Different criteria for the summability of a conjugate Laplace series were given in [5] in the particular case $n = 3$. These criteria are not readily extendable to higher dimensions. Here we show how to obtain the Abel summability of conjugate Laplace series in any dimension.

2. Preliminary. A k-form u is represented in an admissible coordinate system $(x_1, \ldots, x_n)$ as

$$u = \frac{1}{k!} u_{i_1 \ldots i_k} dx_{i_1} \ldots dx_{i_k},$$

where $u_{i_1 \ldots i_k}$ are the components of a k-covector, i.e. the components of a skew-symmetric covariant tensor. We denote the differential, the adjoint and the co-differential operators by d, $*$ and δ, respectively. For details about the theory of differential forms we refer to [6,7].

By $C_k^m(\Omega)$ we denote the space of all k-forms defined in a domain $\Omega \subset \mathbb{R}^n$, whose components are continuously differentiable up to the order m in a coordinate system of class C^{m+1} (and then in every coordinate system of class C^{m+1}). We say that $u \in C_k^1(\Omega)$ and $v \in C_{k+2}^1(\Omega)$ are conjugate if

$$\begin{cases} du = \delta v \\ \delta u = 0, \quad dv = 0. \end{cases} \tag{2}$$

If $n = 2$, $k = 0$, system (2) turns into the Cauchy-Riemann system.

A k-form u is said to be harmonic if

$$(d\delta + \delta d)u = -\Delta u = -\frac{1}{k!}\Delta u_{i_1 \ldots i_k} dx_{i_1} \ldots dx_{i_k} = 0.$$

We note that two conjugate forms are both harmonic forms.

If u is a harmonic function in the unit ball $B = \{x \in \mathbb{R}^n : |x| < 1\}$, we have the expansion

$$u(x) = \sum_{h=0}^{\infty} |x|^h \sum_{k=1}^{N_{h,n}} a_{hk} Y_{hk}\left(\frac{x}{|x|}\right),$$

where $\{Y_{hk}\}$ stands for an orthonormal complete system of spherical harmonics and

$$N_{h,n} \equiv \dim[\mathbb{Y}_{\mathrm{h,n}}(\Sigma)] = \frac{(\mathrm{h}+\mathrm{n}-3)!}{\mathrm{h}!(\mathrm{n}-2)!}(2\mathrm{h}+\mathrm{n}-2), \quad \mathrm{h} \in \mathbb{N},$$

$\mathbb{Y}_{h,n}(\Sigma)$ being the spherical harmonic space of order h in n dimensions.

The trace of u on Σ is given by the expansion

$$\sum_{h=0}^{\infty} \sum_{k=1}^{N_{h,n}} a_{hk} Y_{hk}(x), \quad |x| = 1. \tag{3}$$

If the coefficients a_{hk} are

$$a_{hk} = \int_{\Sigma} Y_{hk} d\mu \qquad (a_{hk} = \int_{\Sigma} f Y_{hk} d\sigma),$$

we say that (3) is the Laplace series of the measure μ (of the function f). In what follows, the term measure means a finite signed measure defined on the Borel sets of Σ.

According to [4,5], we introduce conjugate Laplace series by analogy with the case of trigonometric series. Let us consider the 2-form

$$v(x) = \sum_{h=0}^{\infty} \sum_{k=1}^{N_{h,n}} \frac{a_{hk}}{(h+2)(h+n-2)} dY_{hk}\left(\frac{x}{|x|}\right) \wedge d(|x|^{h+2}). \tag{4}$$

The h-th term of this series is a differential form whose coefficients are harmonic homogeneous polynomials of degree h. It is possible to show that the couple (u, v) satisfies system (2), that means that u and v are conjugate forms. The boundary behaviour of v is determined by the restriction of v and $*v$ on Σ. If the restriction of v exists, it is equal to 0 because of the presence of the term $d(|x|^{h+2})$, while the restriction of $*v$ is (formally at least)

$$\sum_{h=0}^{\infty} \sum_{k=1}^{N_{h,n}} \frac{a_{hk}}{(h+2)(h+n-2)} * \left(dY_{hk}\left(\frac{x}{|x|}\right) \wedge d(|x|^{h+2})\right)\Bigg|_{|x|=1}. \tag{5}$$

We call (5) the series conjugate to the spherical expansion (3). If (3) is a Laplace series, we say that (5) is the Laplace series conjugate to (3).

Let us consider the Laplace series of a measure μ. Arguing as in [5], the series (4) and (5) can be written in a simpler way by means of the Legendre polynomials $P_{h,n}$ as

$$v(x)=\frac{1}{\omega_\Sigma}\sum_{h=1}^{\infty}\frac{N_{h,n}}{h+n-2}|x|^{h-1}\left[\int_\Sigma P'_{h,n}\left(\frac{x}{|x|}\cdot y\right)y_{i_1}x_{i_2}d\mu_y\right]dx_{i_1}dx_{i_2}$$

and

$$\frac{1}{(n-2)!\omega_\Sigma}\sum_{h=1}^{\infty}\frac{N_{h,n}}{h+n-2}\left[\int_\Sigma P'_{h,n}(x\cdot y)\delta^{1\ldots\ldots\ldots n}_{i_1 i_2 j_1\ldots j_{n-2}}y_{i_1}x_{i_2}d\mu_y\right]dx_{j_1}\ldots dx_{j_{n-2}}\Big|_{|x|=1},$$

respectively.

3. Abel summability. We treat now the Abel summability of conjugate Laplace series; this topic is discussed more fully in [8].

Let us consider the series

$$\sum_{h=1}^{\infty}\frac{N_{h,n}}{h+n-2}r^h P'_{h,n}(t). \tag{6}$$

It absolutely converges for $r\in(-1,1)$, $t\in[-1,1]$. Moreover, it uniformly converges for $r\in K\subset(-1,1)$, $t\in[-1,1]$. It is possible to give an integral representation for the series (6). Namely, if $r\in(0,1)$, $t\in[-1,1]$, then

$$\sum_{h=1}^{\infty}\frac{N_{h,n}}{h+n-2}r^h P'_{h,n}(t)=\frac{n}{r^{n-2}}\int_0^r\frac{\rho^{n-2}-\rho^n}{(1+\rho^2-2t\rho)^{\frac{n+2}{2}}}d\rho\equiv J_n(r,t).$$

Setting $r=|x|$ and $t=x\cdot y$, the function $J_n(r,t)$ can be seen like the kernel of conjugate series.

The coefficients $v_{j_1\cdots j_{n-2}}(x)$ of $*v$ satisfy a limit relation, described by the next theorem.

Theorem 1. *Let*

$$v_{j_1\ldots j_{n-2}}(x)\equiv\frac{1}{(n-2)!\omega_\Sigma}\sum_{h=1}^{\infty}\frac{N_{h,n}}{h+n-2}|x|^{h-1}\left[\int_\Sigma P'_{h,n}\left(\frac{x}{|x|}\cdot y\right)\delta^{1\ldots\ldots\ldots n}_{i_1 i_2 j_1\ldots j_{n-2}}y_{i_1}x_{i_2}d\mu_y\right]$$

$(1\le j_k\le n,\ k=1,\ldots,n-2)$, *where* μ *is a measure on* Σ. *If* $x\in\Sigma$ *is a Lebesgue point of* μ, *then*

$$\lim_{\tau\to0^+}\left[v_{j_1\ldots j_{n-2}}((1-\tau)x)-\frac{1}{(n-2)!\omega_\Sigma}\int_{\Sigma\setminus\Sigma_\tau}J_n(1,x\cdot y)\delta^{1\ldots\ldots\ldots n}_{i_1 i_2 j_1\ldots j_{n-2}}y_{i_1}x_{i_2}d\mu_y\right]=0,$$

where $\Sigma_\tau=\{y\in\Sigma:|y-x|<\tau\}$ [1].

[1] We recall that $x\in\Sigma$ is a Lebesgue point for the measure μ if

$$\lim_{\tau\to0^+}\frac{|\mu-f(x)\sigma|(\Sigma_\tau)}{\sigma(\Sigma_\tau)}=0,$$

where $|\cdot|$ is the total variation measure, σ is the $(n-1)$-dimensional Lebesgue measure on Σ and f is the Radon-Nikodym derivative of μ.

Since one can write

$$J_n(1, x\cdot y)\,\delta^{1.........n}_{i_1 i_2 j_1\dots j_{n-2}}\,y_{i_1}\,x_{i_2} = |x-y|^n J_n(1,x\cdot y)\,M_y^{j_1\dots j_{n-2}}\left(\frac{1}{|x-y|^{n-2}}\right),$$

where $M^{j_1\dots j_{n-2}} \equiv \delta^{1.........n}_{i_1 i_2 j_1\dots j_{n-2}}\nu_{i_1}\frac{\partial}{\partial x_{i_2}}$ $(1 \le j_k \le n,\ k=1,\dots,n-2)$ the next statement is obtained by means of some properties involving such tangential operators.

Theorem 2. *If μ is a measure on Σ, the singular integrals*

$$\frac{1}{(n-2)!\omega_\Sigma}\int_\Sigma J_n(1,x\cdot y)\delta^{1.........n}_{i_1 i_2 j_1\dots j_{n-2}} y_{i_1} x_{i_2} d\mu_y$$

$(1 \le j_k \le n,\ k=1,\dots,n-2)$ *do exist almost everywhere on Σ.*

The last two results combine to give the Abel summability of conjugate Laplace series.

Theorem 3. *The conjugate Laplace series of measure μ is Abel summable almost everywhere on Σ and its Abel sum is*

$$(A)\frac{1}{(n-2)!\omega_\Sigma}\sum_{h=1}^{\infty}\frac{N_{h,n}}{h+n-2}\left[\int_\Sigma P'_{h,n}(x\cdot y)\delta^{1.........n}_{i_1 i_2 j_1\dots j_{n-2}} y_{i_1} x_{i_2} d\mu_y\right]dx_{j_1}\dots dx_{j_{n-2}}\Bigg|_{|x|=1}$$
$$= \frac{1}{(n-2)!\omega_\Sigma}\left[\int_\Sigma J_n(1,x\cdot y)\delta^{1.........n}_{i_1 i_2 j_1\dots j_{n-2}} y_{i_1} x_{i_2} d\mu_y\right]dx_{j_1}\dots dx_{j_{n-2}}\Bigg|_{|x|=1}.$$

R E F E R E N C E S

1. Zygmund A. Trigonometric Series. *Cambridge University Press*, 1979.
2. Muckenhoupt B., Stein E.M. Classical expansions and their relation to conjugate harmonic functions. *Trans. Amer. Math. Soc.*, **118**, (1965), 17-92.
3. Li Z. Conjugate Jacobi series and conjugate functions. *J. Approx. Theory*, **86**, (1996), 179-196.
4. Cialdea A. The Brothers Riesz theorem in $\mathbb{R}^n$ and Laplace series. *Mem. Differential Equations Math. Phys.*, **12**, (1997), 42-49.
5. Cialdea A. The summability of conjugate Laplace series on the sphere. *Acta Sci. Math. (Szeged)*, **65**, (1999), 93-119.
6. Fichera G. Spazi lineari di k-misure e di forme differenziali. *Proc. Intern. Sympos. Linear Spaces, Jerusalem 1960, Jerusalem Acad. Press; Pergamon Press*, (1961), 175-226.
7. Flanders H. Differential Forms with Applications to the Physical Sciences. *Dover Publications Inc., New York*, 1989.
8. Caramuta P., Cialdea A., Silverio F. The Abel summability of conjugate Laplace series of measures. (submitted).

Received 23.05.2014; revised 22.11.2014; accepted 29.12.2014.

Authors' address:

P. Caramuta, A. Cialdea, F. Silverio
Department of Mathematics, Computer Science and Economics
University of Basilicata
V.le dell'Ateneo Lucano, 10 85100 Potenza
Italy
E-mail: pietro.caramuta@unibas.it
cialdea@email.it, silveriofrancesco@hotmail.com