Reports of Enlarged Session of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 28, 2014

PRECISE EXPONENTIAL MR-GROUPS

Amaglobeli M.

Abstract. The category of exponential MR-groups for an associative ring R with unity is defined in [1]. The present paper is devoted to the study of partial MR-exponential groups which are isomorphically embedded in their tensor completion over the ring R. The key to its understanding is the notion of tensor completion introduced in [1]. As a consequence, the description of free MR-groups in the language of group constructions is obtained.

Keywords and phrases: Exponential R-group, Lyndon R-group, MR-group, tensor completion.

AMS subject classification: 20B07.

Let R be any associative ring with identity. Myasnikov and Remeslennikov [1] introduced a new category of exponential R-groups as a natural generalization of the notion of an R-module to the noncommutative case. Below, we recall the basic definitions borrowed from [1], [2].

Let $\mathcal{L}_{gr} = \langle \cdot\, , ^{-1}, e\rangle$ be the group language (signature); here $\cdot$ denotes the binary operation of multiplication, $^{-1}$ denotes the unary operation of inversion, and e is a constant symbol for the identity element of the group.

We enrich the group language to the language $\mathcal{L}^*_{gr} = \mathcal{L}_{gr} \cup \{f_\alpha(x)|\ \alpha \in R\}$, where $f_\alpha(x)$ is the unary algebraic operation.

Definition 1. A Lyndon R-group is a set G on which the operations $\cdot$, $^{-1}$, e, and $\{f_\alpha(x)|\ \alpha \in R\}$ are defined and the following axioms hold:

(i) the group axioms;

(ii) for all $g, h \in G$ and elements $\alpha, \beta \in R$,

$$g^1 = g, \quad g^0 = e, \quad e^\alpha = e; \tag{1}$$

$$g^{\alpha+\beta} = g^\alpha \cdot g^\beta, \quad g^{\alpha\beta} = (g^\alpha)^\beta; \tag{2}$$

$$(h^{-1}gh)^\alpha = h^{-1}g^\alpha h. \tag{3}$$

For brevity, in the formulas expressing the axioms, we write $f_\alpha(g)$ instead of g^α for $g \in G$ and $\alpha \in R$.

Let $\mathrm{LG}(R)$ denote the category of all Lyndon R-groups. Since the axioms given above are universal axioms of the language $\mathcal{L}^*_{gr}$, the general theorems of inversal algebra allow us to consider the variety of R-groups. R-homomorphisms, R-isomorphisms, free R-groups, and so on.

MR**-exponential groups.** There exist Abelian Lyndon R-groups which are not R-modules (see [3], where the structure of a free Abelian R-group was studied in detail). The authors of [1] augmented Lyndon's axioms by the additional axiom (quasi-identity):

$$\text{(MR)} \qquad \forall\, g, h \in G,\ \ \alpha \in R\ \ [g,h] = 1 \Longrightarrow (gh)^\alpha = g^\alpha h^\alpha. \tag{4}$$

Definition 2. An MR-group is a group G on which the operations g^α are defined for all $g \in G$ and $\alpha \in R$ so that axioms (1)–(4) hold.

Let MG(R) denote the class of all R-exponential groups with axioms (1)–(4). Clearly, this class is a quasi-variety in the language $\mathcal{L}^*_{gr}$, and free MR-groups, MR-homomorphisms, and so on are defined; moreover, each Abelian MR-group is an R-module, and vice versa.

Most of natural examples of exponential groups belong to the class MG(R):

1) An arbitrary group is a $\mathbb{Z}$-group;

2) An Abelian divisible group from LG($\mathbb{Q}$) is an MG($\mathbb{Q}$)-group;

3) A group of the period m is a $\mathbb{Z}/m\mathbb{Z}$-group;

4) A module over the ring R is an Abelian MR-group;

5) Free Lyndon R-groups are MR-groups;

6) The exponential nilpotent R-groups over the binomial ring R introduced by F. Hall in [4] are MR-groups;

7) An arbitrary pro-p-group is a $\mathbb{Z}_{p^\infty}$-group over a ring of integer p-adic numbers $\mathbb{Z}_{p^\infty}$.

Definition 3. A homomorphism of R-groups $\varphi : G \to H$ is called an *R-homomorphism* if

$$\varphi(g^\alpha) = \varphi(g)^\alpha, \ \ g \in G, \ \ \alpha \in R.$$

For the basic definitions in the category MG(R) and the results an these groups we refer the reader to [1], [2], [6].

For the completeness of our discussion, we give here the definition of the notion of tensor completion.

Definition 4. Let G be an R-group and $\mu : R \to S$ a homomorphism of rings. Then a S-group G^S will be called a *tensor S-completion* of the group G if G^S satisfies the following universal property:

(1) there exists an R-homomorphism $\lambda : G \to G^S$ such that $\lambda(G)$ S-generates G^S, i.e. $\langle \lambda(G) \rangle_S = G^S$;

(2) for any S-group H and any R-homomorphism $\varphi : G \to H$ coordinated with μ (i.e., $\varphi(g^\alpha) = \varphi(g)^{\mu(\alpha)}$) there exists a S-homomorphism $\psi : G^S \to H$, rendering the following diagram commutative:

$$\begin{array}{ccc} G & \xrightarrow{\lambda} & G^S \\ {\scriptstyle\varphi}\downarrow & \swarrow{\scriptstyle\psi} & \\ H & & \end{array} \quad , \quad \lambda\psi = \varphi.$$

Note that if G is an Abelian R-group, then $G^S \cong G \underset{R}{\otimes} S$ is a tensor product of an R-module G by a ring S.

In [1], it is proved that for any MR-group G and any homomorphism $\mu : R \to S$ the tensor completion G^S exists always and it is unique to within an isomorphism.

In applications, $\mu : R \to S$ is most often be an inclusion map of rings, but in this case the homomorphism $\lambda : G \to G^S$ is not necessarily an inclusion map. Since in the Abelian case the group G^S is obtained by the operation of tensor product of R-module G a ring S, corresponding examples are in many books on commutative algebra and homology theory. The following proposition deseribes the situation where λ is an inclusion map.

Definition 5. We say that an R-group *is approximated by a S-group* with respect to a homomorphism μ if for any $e \neq g \in G$ there exists an MR-homomorphism $\varphi_g : G \to H$ coordinated with μ such that $\varphi_g(g) \neq e$.

Proposition 1 ([1]). *Let an* MR-*group G be approximated with respect to a homomorphism μ. Then the homomorphism $\lambda : G \to G^S$ is an inclusion map.*

In what follows we investigate the notion of *tensor completion precision.* It is convenient to construct a tensor completion of a given group step by step, defining powers gradually. This involves the notion of a partial R-group. Some group operations on an MR-group involve partial MR-groups. Let R be a ring, let G be a group.

Definition 6. We call the group G a *partial* MR-*group* if exponentiation is defined for some pairs (g, α), but not necessarily defined for all pairs and if a part of the equation in the axioms (1)–(4) of the definition of an exponential group is true. The class of partial MR-exponential groups is denoted by $\mathcal{P}_R$.

Example. let be a subgroup R of the ring S. Then any MR-group is a partial MS-group.

In the remainder of the paper it is assumed that the ring R as a subring contains the ring of integer numbers $\mathbb{Z}$.

Let G be a partial R-group, i.e. $G \in \mathcal{P}_R$.

Definition 7. We say that the group G is *precise with respect to the ring R* if the canonical mapping $\lambda : G \to G^R$ is an embedding.

Definition 8. We say that the group G is *precise* if it is precise with respect to any ring containing $\mathbb{Z}$.

In [5], G. Baumslag proved that if G is a free group and R is a field of rational numbers $\mathbb{Q}$, then G is a precise group. In [2], R. Lyndon proved that if G is a free group, $R = \mathbb{Q}$ or $R = \mathbb{Z}[z_1, \dots, x_n]$ is a ring of integer polynomials, then G is a precise group.

We will prove precision for a wider class of groups and a wider class of rings. Let us introduce the needed definitions.

Let R be a ring, $\mathcal{P}_R^0$ be the category of partial R-groups. By definition, a group G from $\mathcal{P}_R$ belongs to $\mathcal{P}_R^0$ if:

1) for any maximal Abelian subgroup M and any $x \notin M$, the intersection $M \cap M^x = 1$;

2) the canonical homomorphism $i : M \to M \underset{R}{\otimes} R$ is an embedding.

Here we use the notion of a partial tensor product which is intoroduced and studied in [1].

The following theorem is the basic one.

Theorem 1 ([6]). *Let $\mathbb{Z}$ be a subring of the ring R and the group $G \in \mathcal{P}_R^0$; also, G and R^+ (an additive group of the ring R) not contain elements of order 2. Then the group G^R is precise, i.e. the canonical homomorphism $\lambda : G \to G^R$ is an embedding.*

This theorem provides a sufficient condition for the tensor completion to be precise. Note that condition 1) from the definition of the class $\mathcal{P}_R^0$ is also a necessary one.

The proof of this theorem employs the technique of tensor completion construction based on the construction of a free product of groups with an adjoint subgroup and the technique of combinatorial theory of groups.

Theorem 2 ([6]). *The class $\mathcal{P}_R^0$ contains free groups closed with respect to direct limits and free products.*

Thus we see that this theorem generalizes the results of G. Baumslag and R. Lyndon.

Theorem 3 ([6]). *The tensor completion of an abstract free group $F(X)^R$ is an R-free group $F_R(X)$ with base X.*

Let us formulate the corollary of Basic Theorem 1 and Theorem 2.

Corollary. Let R be the ring containing $\mathbb{Z}$ as a subring. Then the free group $F(X)$ is precise with respect to the ring R. In other words, $F(X)$ is a subgroup of $F_R(X)$.

REFERENCES

1. Myasnikov A.G., Remeslennikov V.N. Degree groups. I. Foundations of the theory and tensor completions. (Russian) *Sibirsk. Mat. Zh.*, **35**, 5 (1994), 1106-1118. English translation: *Siberian Math. J.*, **35**, 5 (1994), 986-996.

2. Lyndon R.C. Groups with parametric exponents. *Trans. Amer. Math. Soc.* **96**, (1960), 518-533.

3. Baumslag G. Free Abelian X-groups. *Illinois J. Math.*, **30**, 2 (1986), 235-245.

4. Hall P. The Edmonton Notes on Nilpotent Groups. *Queen Mary College, London*, 1957.

5. Baumslag G. On free $\mathcal{D}$-groups. *Comm. Pure Appl. Math.*, **18**, (1965), 25-30.

6. Amaglobeli M. Power groups. *J. Math. Sci.*, **186**, 6 (2012), 811-865.

Received 12.05.2014; revised 11.11.2014; accepted 29.12.2014.

Author's address:

M. Amaglobeli
Iv. Javakhishvili Tbilisi State University
2, University St., Tbilisi 0186
Georgia
E-mail: mikheil.amaglobeli@tsu.ge