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SOME BASIC BOUNDARY VALUE PROBLEMS FOR THE HIERARCHICAL MODELS

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Abstract. In the present paper, by means of Vekua's method, the system of differential equations for thin and shallow shells is obtained, when on upper and lower face surfaces displacements are assumed to be known.

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I. Vekua constructed several versions of the refined linear theory of thin and shallow shells, containing the regular processes by means of the method of reduction of 3-D problems of elasticity to 2-D ones, when on upper and lower face surfaces stress-vectors are assumed to be known [1-2].

The equilibrium equation of the continuous medium and stress-strain relations (Hooke's law) can be written in the form

$$\frac{1}{\sqrt{g}}\partial_i\Big(\sqrt{g}\boldsymbol{\sigma}^i\Big)+\boldsymbol{\Phi}=0, \tag{1}$$

$$\boldsymbol{\sigma}^i=\lambda\Big(\boldsymbol{R}^j\partial_j\boldsymbol{u}\Big)\boldsymbol{R}^i+\mu\Big(\boldsymbol{R}^i\partial_j\boldsymbol{u}\Big)\boldsymbol{R}^j+\mu\Big(\boldsymbol{R}^i\boldsymbol{R}^j\Big)\partial_j\boldsymbol{u}, \tag{2}$$
$$(i,j=1,2,3)$$

where $\partial_i=\dfrac{\partial}{\partial x^i}$, x^i are curvilinear coordinates, g is the discriminant of the metric tensor of the space, $\boldsymbol{\Phi}$ is the volume force, $\boldsymbol{\sigma}^i$ are contravariant stress vectors, λ and μ are Lame's constants, $\boldsymbol{R}^i$ and $\boldsymbol{R}_i$ are covariant and contravariant base vectors of the space and $\boldsymbol{u}$ is the displacement vector.

To construct the theory of shells, we use the more convenient coordinate system which is normally connected with the midsurface of the shell [1-2].

For thin or shallow shells we can write [1-2]

$$\boldsymbol{R}_\alpha\cong\boldsymbol{r}_\alpha,\quad \boldsymbol{R}^\alpha\cong\boldsymbol{r}^\alpha,\quad \boldsymbol{R}^3=\boldsymbol{R}_3=\boldsymbol{n},\quad g\cong a,$$

where $\boldsymbol{r}_\alpha$ and $\boldsymbol{r}^\alpha$ are covariant and contravariant base vectors of the midsurface, $\boldsymbol{n}$ is the unit vector of the normal of the midsurface, a is the discriminant of the metric tensor of the midsurface.

Multiplying both sides of equations (1) and (2) by Legendre polynomials $P_m\left(\frac{x^3}{h}\right)$ and then integrating with respect to x^3 from $-h$ to h we obtain the equivalent infinite system of 2-D equations

$$\frac{1}{\sqrt{a}}\frac{\sqrt{a}\overset{(m)}{\boldsymbol{\sigma}}{}^{\alpha}}{\partial x^\alpha}+\frac{2m+1}{h}\left(\overset{(m+1)}{\boldsymbol{\sigma}}{}^{3}+\overset{(m+3)}{\boldsymbol{\sigma}}{}^{3}+...\right)+\overset{(m)}{\boldsymbol{\Phi}}=0, \tag{3}$$

$$\begin{aligned}
&\overset{(m)}{\boldsymbol{\sigma}}{}^{\alpha} = \lambda\left[\boldsymbol{r}^{\beta}\partial_{\beta}\overset{(m)}{\boldsymbol{u}} - \frac{2m+1}{h}\left(\overset{(m-1)}{u}{}_{3} + \overset{(m-3)}{u}{}_{3} + \cdots\right)\right]\boldsymbol{r}^{\alpha}\\
&+\left[\left(\boldsymbol{r}^{\alpha}\partial_{\beta}\overset{(m)}{\boldsymbol{u}}\right)\boldsymbol{r}^{\beta} + \partial^{\alpha}\overset{(m)}{\boldsymbol{u}} - \frac{2m+1}{h}\left(\overset{(m-1)}{u}{}_{\alpha} + \overset{(m-3)}{u}{}_{\alpha} + \cdots\right)\boldsymbol{n}\right] + \overset{(m)}{\boldsymbol{F}}{}^{\alpha},\\
&\overset{(m)}{\boldsymbol{\sigma}}{}^{3} = \lambda\left(\boldsymbol{r}^{\beta}\partial_{\beta}\overset{(m)}{\boldsymbol{u}}\right)\boldsymbol{n} - (\lambda+\mu)\frac{2m+1}{h}\left(\overset{(m-1)}{u}{}_{\alpha} + \overset{(m-3)}{u}{}_{\alpha} + \cdots\right)\boldsymbol{n}\\
&+\mu\left[\left(\boldsymbol{n}\partial_{\beta}\overset{(m)}{\boldsymbol{u}}\right)\boldsymbol{r}^{\beta} - \frac{2m+1}{h}\left(\overset{(m-1)}{\boldsymbol{u}}{}_{3} + \overset{(m-3)}{\boldsymbol{u}}{}_{3} + \cdots\right)\right] + \overset{(m)}{\boldsymbol{F}}{}^{3},
\end{aligned} \tag{4}$$

$$(\alpha = 1,2;\ m = 0,1,...)$$

where

$$\left(\overset{(m)}{\boldsymbol{\sigma}}{}^{i}, \overset{(m)}{\boldsymbol{u}}, \overset{(m)}{\boldsymbol{\Phi}}\right) = \frac{2m+1}{2h}\int_{-h}^{h}(\boldsymbol{\sigma}^{i}, \boldsymbol{u}, \boldsymbol{\Phi})P_m\left(\frac{x_3}{h}\right)dx_3,$$

$$\overset{(m)}{\boldsymbol{F}}{}^{\alpha} = \frac{2m+1}{2h}\left[\lambda\left(\overset{(+)}{u}{}^{3} - (-1)^m\overset{(-)}{u}{}^{3}\right)\boldsymbol{r}^{\alpha} + \mu\left(\overset{(+)}{u}{}^{\alpha} - (-1)^m\overset{(-)}{u}{}^{\alpha}\right)\boldsymbol{n}\right],$$

$$\overset{(m)}{\boldsymbol{F}}{}^{3} = \frac{2m+1}{2h}\left[(\lambda+\mu)\left(\overset{(+)}{u}{}^{3} - (-1)^m\overset{(-)}{u}{}^{3}\right)\boldsymbol{n} + \mu\left(\overset{(+)}{\boldsymbol{u}} - (-1)^m\overset{(-)}{\boldsymbol{u}}\right)\right],$$

$$\overset{(\pm)}{\boldsymbol{u}} = \boldsymbol{u}(x^1, x^2, \pm h).$$

x^1, x^2 are the Gaussian parameters of the midsurfaces, $x^3 = x_3$ is the thickness coordinate and is the semi-thickness. So, we get the equivalent to (1), (2) infinite system.

Then we consider $N = 0$ approximation for plates. In other words, in the previous equations it is assumed that

$$\overset{(m)}{\boldsymbol{u}} = 0, \quad \overset{(m)}{\boldsymbol{\sigma}}{}^{i} = 0, \quad if\ \ m > 0.$$

From (3) and (4) we obtain the following complex writing the system of equations of equilibrium

$$\begin{aligned}
&\mu\Delta\overset{(0)}{u}{}_{+} + 2(\lambda+\mu)\partial_{\bar z}\overset{(0)}{\theta} = \overset{(0)}{\Psi}{}_{+}\\
&\mu\Delta\overset{(0)}{u}{}_{3} = \overset{(0)}{\Psi}{}_{3},
\end{aligned} \tag{5}$$

$$\overset{(0)}{\Psi}{}_{+} = -\overset{(0)}{\Phi}{}_{+} - \partial_z(\overset{(0)}{F}_{11} - \overset{(0)}{F}_{22} + 2i\overset{(0)}{F}_{12}) - \partial_{\bar z}(\overset{(0)}{F}_{11} + \overset{(0)}{F}_{22}),$$

$$\overset{(0)}{\Psi}{}_{3} = \overset{(0)}{\Phi}{}_{3} - \partial_z\overset{(0)}{F}_{+3} - \partial_{\bar z}\overset{(0)}{\bar F}_{+3},$$

where

$$z = x_1 + ix_2,\ \ \partial_z = \frac{1}{2}(\partial_1 - i\partial_2),\ \ \partial_{\bar z} = \frac{1}{2}(\partial_1 + i\partial_2),\ \ \Delta = 4\partial^2_{z\bar z},$$

$$\overset{(0)}{u}{}_{+} = \overset{(0)}{u}{}_{1} + i\overset{(0)}{u}{}_{2},\ \ \overset{(0)}{\Phi}{}_{+} = \overset{(0)}{\Phi}{}_{1} + i\overset{(0)}{\Phi}{}_{2},\ \ \overset{(0)}{F}_{+3} = \overset{(0)}{F}_{13} + i\overset{(0)}{F}_{23}.$$

The general solutions of system (5) have the following forms [3]:

$$\begin{aligned} 2\mu \overset{(0)}{u}_{+} &= \varkappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)}, \\ 2\mu \overset{(0)}{u}_{3} &= f(z) + \overline{f(z)}, \end{aligned} \tag{6}$$

where $\varphi(z)$, $\psi(z)$ and $f(z)$ are analytic functions of z.

The particular solutions of equations (5) have the following forms:

$$\overset{(0)}{u}_{+p} = \frac{1}{\pi}\frac{\lambda+3\mu}{4\mu(\lambda+2\mu)}\iint\limits_{D} \overset{(0)}{\Psi}_{+}(\xi,\eta)\ln|\zeta - z|d\xi d\eta$$

$$-\frac{1}{\pi}\frac{\lambda+\mu}{8\mu(\lambda+2\mu)}\iint\limits_{D} \overline{\overset{(0)}{\Psi}}_{+}(\xi,\eta)\frac{\zeta - z}{\bar{\zeta}-\bar{z}}d\xi d\eta,$$

$$\overset{(0)}{u}_{3p} = \frac{1}{4\pi\mu}\iint\limits_{D} \overset{(0)}{\Psi}_{3}(\xi,\eta)\ln|\zeta - z|d\xi d\eta,$$

where $\zeta = \xi + i\eta$.

Let us consider the case, when the middle surface of the body is the circle with radius R, Φ_i are equal to P_i, where $P_i = const$, $\boldsymbol{u}(x^1, x^2, \pm h) = 0$.

Boundary conditions have the following form:

$$\overset{(0)}{u}_{r} + i\overset{(0)}{u}_{\theta} = i\overset{(0)}{u}_{+}\frac{d\bar{z}}{ds} = 0, \quad r = R,$$

$$\overset{(0)}{u}_{3} = 0, \quad r = R.$$

The solution of this problem is [3, 4]:

$$2\mu\overset{(0)}{u}_{+} = \frac{\mu}{\lambda+3\mu}(z\bar{z} - R^2)P_{+},$$

$$2\mu\overset{(0)}{u}_{3} = (z\bar{z} - R^2)P_3,$$

where $P_{+} = P_1 + iP_2$.

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