Reports of Enlarged Session of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 28, 2014

ABOUT ONE BOUNDARY VALUE PROBLEM FOR NONLINEAR NON-SHALLOW SPHERICAL SHELLS

Gulua B.

Abstract. In the present paper, using the method of I. Vekua, the three dimensional problems of the nonlinear theory of elasticity are reduced to the two dimensional problems of non-shallow spherical shells. Using the method of the small parameter, approximate solutions of these equations are constructed. One boundary value problems are solved for the approximation of order $N = 0$.

Keywords and phrases: Non-shallow shell, displacement vector, stress vector.

AMS subject classification: 74K25, 74B20.

I. Vekua has constructed the refined theory of shallow shells [1],[2]. This method for non-shallow shells in case of the geometrical and physical non-linear theory was generalized by T.Meunargia [3],[4].

In the present paper we consider the system of equilibrium equations of the two-dimensional geometrically nonlinear non-shallow spherical shells which are obtained from the three-dimensional problems of the theory of elasticity for isotropic and homogeneous shell by the method of I. Vekua.

A complete system of equations of the three-dimensional nonlinear theory of elasticity can be written as:

$$\frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}\vec{T}^i}{\partial x^i}+\vec{\Phi}=0, \tag{1}$$

$$\begin{aligned}\vec{T}^i=&\lambda\left(\vec{R}^j\partial_j\vec{u}+\frac{1}{2}\partial^j\vec{u}\partial_j\vec{u}\right)\left(\vec{R}^i+\partial^i\vec{u}\right)\\&+\mu\left(\vec{R}^i\partial^j\vec{u}+\vec{R}^j\partial^i\vec{u}+\partial^i\vec{u}\partial^j\vec{u}\right)\left(\vec{R}_j+\partial_j\vec{u}\right),\end{aligned} \tag{2}$$

where x^1, x^2 and x^3 are curvilinear coordinates, g is the discriminant of the metric tensor of the space, $\vec{T}^i$ are contravariant stress vectors, $\vec{\Phi}$ is an external force, λ and μ are Lame's constants, $\vec{R}_i$ and $\vec{R}^i$ are covariant and contravariant base vectors of the space and $\vec{u}$ is the displacement vector.

From (1) the system of equilibrium equations of the two-dimensional geometrically nonlinear and non-shallow spherical shells may be written in the following form (approximation N=0):

$$\begin{aligned}&\nabla_\alpha\sigma^{\alpha\beta}-\varepsilon\sigma^{\alpha3}+F^\beta=0,\quad\beta=1,2\\&\nabla_\alpha\sigma^{\alpha3}+\varepsilon\sigma^\alpha_\beta+F^3=0,\end{aligned} \tag{3}$$

where $\vec{\sigma}^i=\sigma^{i\beta}\vec{r}_\beta+\sigma^{i3}\vec{n}$, $\varepsilon=\dfrac{h}{\rho}$, $2h$ is the thickness of the shell, ρ is the radius of the middle surface of the spherical shell, ∇_α are covariant derivatives and

$$\vec{F} = \overset{(0)}{\vec{\Phi}} + \frac{1}{2h}\left[(1+\varepsilon)^2 \overset{(+)}{\vec{T}}_3 - (1-\varepsilon)^2 \overset{(-)}{\vec{T}}_3\right],$$

$$\left(\sigma_{ij},\ \overset{(0)}{\vec{\Phi}}\right) = \frac{1}{2h}\int\limits_{-h}^{h}\left(1+\frac{x_3}{R}\right)^2 (T_{ij},\ \vec{\Phi})dx_3.$$

$$\vec{T}_3(x_1, x_2, \pm h) = \overset{(\pm)}{\vec{T}}_3.$$

Hooke's law has the form:

$$\begin{aligned}
&\vec{\sigma}^1 = \lambda(\vec{r}^\beta\partial_\beta\vec{u})\vec{r}^1 + \mu\left[2(\vec{r}^1\partial^1\vec{u})\vec{r}_1 + (\vec{r}^1\partial^2\vec{u})\vec{r}_2 + (\vec{r}^2\partial^1\vec{u})\vec{r}_2 + (\vec{n}\partial^1\vec{u})\vec{n}\right]\\
&+\left(1+\frac{\varepsilon^2}{3}+\frac{\varepsilon^4}{5}+\cdots\right)\left\{\lambda\left[(\vec{r}^\beta\partial_\beta\vec{u})\partial^1\vec{u} + \frac{1}{2}(\partial^\beta\vec{u}\partial_\beta\vec{u})\vec{r}^1\right] + \mu[2(\vec{r}^1\partial^1\vec{u})\partial_1\vec{u}\right.\\
&\left.+(\partial^1\vec{u}\partial_1\vec{u})\vec{r}^1 + (\vec{r}^1\partial^2\vec{u})\partial_2\vec{u} + (\vec{r}^2\partial^1\vec{u})\partial_2\vec{u} + (\partial^2\vec{u}\partial^1\vec{u})\vec{r}_2]\right\}\\
&+\left(1+\varepsilon^2+\varepsilon^4+\cdots\right)\left\{\frac{\lambda}{2}(\partial^\beta\vec{u}\partial_\beta\vec{u})\partial^1\vec{u} + \mu(\partial^1\vec{u}\partial^1\vec{u})\partial_1\vec{u} + \mu(\partial^2\vec{u}\partial^1\vec{u})\partial_2\vec{u}\right\},\\
&\vec{\sigma}^2 = \lambda(\vec{r}^\beta\partial_\beta\vec{u})\vec{r}^2 + \mu\left[2(\vec{r}^2\partial^2\vec{u})\vec{r}_2 + (\vec{r}^2\partial^1\vec{u})\vec{r}_1 + (\vec{r}^1\partial^2\vec{u})\vec{r}_1 + (\vec{n}\partial^2\vec{u})\vec{n}\right]\\
&+\left(1+\frac{\varepsilon^2}{3}+\frac{\varepsilon^4}{5}+\cdots\right)\left\{\left[\lambda(\vec{r}^\beta\partial_\beta\vec{u})\partial^2\vec{u} + \frac{1}{2}(\partial^\beta\vec{u}\partial_\beta\vec{u})\vec{r}^2\right] + \mu[2(\vec{r}^2\partial^2\vec{u})\partial_2\vec{u}\right.\\
&\left.+(\partial^2\vec{u}\partial_2\vec{u})\vec{r}^2 + (\vec{r}^2\partial^1\vec{u})\partial_1\vec{u} + (\vec{r}^1\partial^2\vec{u})\partial_1\vec{u} + (\partial^2\vec{u}\partial^1\vec{u})\vec{r}_1\right\}\\
&+\left(1+\varepsilon^2+\varepsilon^4+\cdots\right)\left\{\frac{\lambda}{2}(\partial^\beta\vec{u}\partial_\beta\vec{u})\partial^2\vec{u} + \mu(\partial^2\vec{u}\partial^2\vec{u})\partial_2\vec{u} + \mu(\partial^2\vec{u}\partial^1\vec{u})\partial_1\vec{u}\right\},\\
&\vec{\sigma}^3 = \lambda\left[\vec{r}^\beta\partial_\beta\vec{u} + \frac{1}{2}\partial^\beta\vec{u}\partial_\beta\vec{u}\right]\vec{n} + \mu(\vec{n}\partial^\beta\vec{u})(\vec{r}_\beta + \partial_\beta\vec{u}).
\end{aligned} \tag{4}$$

To find components of the displacements vector and stress tensor, we take the following series of expansions with respect to the small parameter ε[5]

$$(u_i,\ \vec{\sigma}_i,\ F_i) = \sum_{k=1}^{\infty}(\overset{(k)}{u}_i, \overset{(k)}{\vec{\sigma}}_i, \overset{(k)}{F_i})\varepsilon^k, \tag{5}$$

Substituting the above expansions into relations (3), (4) and then equalizing the coefficients of expansions for ε^n, we obtain the following system of equations:

$$4\mu\frac{\partial}{\partial\bar{z}}\left(\frac{1}{\Lambda}\frac{\partial \overset{(k)}{u}_+}{\partial\bar{z}}\right) + 2(\lambda+\mu)\frac{\partial\overset{(k)}{\theta}}{\partial\bar{z}} = \overset{(k)}{X}_+\left(\overset{(0)}{u}_i, ..., \overset{(k-1)}{u}_i\right) \tag{6}$$

$$\mu\nabla^2\overset{(k)}{u}_3 = \overset{(k)}{X}_3\left(\overset{(0)}{u}_i, ..., \overset{(k-1)}{u}_i\right), \tag{7}$$

where $x^1 = tg\frac{\theta}{2}cos\varphi$, $x^2 = tg\frac{\theta}{2}sin\varphi$, $\left(z = x^1 + ix^2, \Lambda = \frac{4\rho^2}{(1+z\bar{z})^2}, \nabla^2 = \frac{4}{\Lambda}\frac{\partial^2}{\partial z\partial\bar{z}}\right)$, are the isometric coordinates on the shell midsurface of a spherical shell,

$$\overset{(k)}{u}_+ = \overset{(k)}{u}_1 + i\overset{(k)}{u}_2,\quad \overset{(k)}{\theta} = \frac{1}{\Lambda}\left(\frac{\partial\overset{(k)}{u}_+}{\partial z} + \frac{\partial\overset{(k)}{\overline{u}}_+}{\partial\bar{z}}\right).$$

Introducing the well-known differential operators

$$\partial_z = \frac{1}{2}\left(\frac{\partial}{\partial x^1} - i\frac{\partial}{\partial x^2}\right), \quad \partial_{\bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x^1} + i\frac{\partial}{\partial x^2}\right).$$

$\overset{(k)}{X}_+$ and $\overset{(k)}{X}_3$ are expressed by $\overset{(0)}{u}_+, \overset{(0)}{u}_3, ..., \overset{(k-1)}{u}_+, \overset{(k-1)}{u}_3$ and it is assumed that they are already found.

Simple calculations show that general solutions of system (6) and (7) can be represented by means of three analytic functions of z in the form

$$\overset{(k)}{u}_+ = -\frac{\varkappa}{\pi}\iint\limits_D \frac{\Lambda(\zeta,\overline{\zeta})\varphi'(\zeta)d\xi d\eta}{\overline{\zeta}-\overline{z}} + \left(\frac{1}{\pi}\iint\limits_D \frac{\Lambda(\zeta,\overline{\zeta})d\xi d\eta}{\overline{\zeta}-\overline{z}}\right)\overline{\varphi'(z)} \tag{8}$$

$$-\overline{\psi(z)} + \frac{1}{8\mu h^2}\frac{\lambda+\mu}{\lambda+2\mu}\frac{1}{\pi}\iint\limits_D \frac{\overset{(k)}{F}_+(\zeta,\overline{\zeta})d\xi d\eta}{\overline{\zeta}-\overline{z}}$$

$$\overset{(k)}{u}_3 = f(z) + \overline{f(z)} - \frac{2}{\pi}\iint\limits_D \overset{(k)}{X}_3 ln|\zeta - z|d\xi d\eta, \tag{9}$$

where $\varphi'(z), f(z)$ and $\psi(z)$ are analytic functions of $z = x_1 + ix_2 \in D$, and $\zeta = \xi + i\eta$. Further,

$$\overset{(k)}{F}_+(z,\overline{z}) = -\frac{1}{\pi}\iint\limits_D \left(\frac{\overline{\overset{(k)}{X}}}{\overline{\zeta}-\overline{z}} - \frac{\varkappa\overset{(k)}{X}}{\zeta - z}\right)d\xi d\eta, \quad \left(\varkappa = \frac{\lambda+3\mu}{\lambda+\mu}\right).$$

D is the domain of the plane Ox_1x_2 onto which the midsurface S of the shell Ω is mapped topologically.

Here we present a general scheme of solution of boundary problems when the domain D is a circle of radius r_0.

The second boundary problem (in displacements) for any k takes the form

$$\overset{(k)}{u}_+|_{r_0} = \Big(-\frac{\varkappa}{\pi}\iint\limits_D \frac{\Lambda(\zeta,\overline{\zeta})\varphi'(\zeta)d\xi d\eta}{\overline{\zeta}-\overline{z}}$$

$$+\left(\frac{1}{\pi}\iint\limits_D \frac{\Lambda(\zeta,\overline{\zeta})d\xi d\eta}{\overline{\zeta}-\overline{z}}\right)\overline{\varphi'(z)} - \overline{\psi(z)}\Big)_{r_0} = \overset{(k)}{G}_+, \quad (|z| = r_0) \tag{10}$$

$$\overset{(k)}{u}_3|_{r_0} = f(z) + \overline{f(z)}|_{r_0} = \overset{(k)}{G}_3. \tag{11}$$

where $\overset{(k)}{G}_+$ and $\overset{(k)}{G}_3$ are the known values containing solutions $\overset{(0)}{u}_i, \overset{(1)}{u}_i, ..., \overset{(k-1)}{u}_i,$ $(i = 1, 2, 3)$ of the previous approximations.

Next $\varphi'(z)$, $\psi(z)$ and $\overset{(k)}{G}_{+}$ are expanded in power series of the type

$$\varphi'(z) = \sum_{n=0}^{\infty} a_n z^n, \quad \psi(z) = \sum_{n=1}^{\infty} b_n z^n, \quad \overset{(k)}{G}_{+} = \sum_{n=\infty}^{\infty} A_k e^{ik\theta}.$$

Assuming that the above-mentioned series for $\varphi'(z)$ and $\psi(z)$ converge not only inside the circle $|z| = r_0$, but also on the circumference $|z| = r_0$ and then substituting these expansions into (9) and comparing the coefficients for $e^{ik\theta}$ we obtain

$$a_k = \frac{1}{\varkappa} \frac{r_0^{k+1}}{\alpha_k(r_0)} A_{k+1}, \quad k \geq 1, \quad b_k = -\Big(\frac{\overline{A}_{-k}}{r_0^k} + \frac{a_0}{\varkappa} \frac{r_0^{k+2}}{\alpha_{k+1}} A_{k+2}\Big), \quad k \geq 0,$$

$$a_0 = \frac{r_0}{\alpha_0} \frac{\varkappa A_1 + \overline{A}_1}{\varkappa^2 - 1},$$

where

$$\alpha_k = -\frac{r_0^2}{1 + r_0^2} + (-1)^{k-1} k \Big[ln(1 + r_0^2) - \sum_{s=1}^{k-1} \frac{(-r_0^2)^s}{s} \Big].$$

A solution of the boundary problem (11) is representable in the form of the Poisson integral

$$\overset{(k)}{u}_{3}(r, \varphi) = \frac{1}{2\pi} \int_0^{2\pi} \overset{(k)}{G}_{3} \frac{r_0^2 - r^2}{r^2 - 2r_0 r \cos(\psi - \theta) + r_0^2} d\psi.$$

Acknowledgement. The designated project has been fulfilled by financial support of the Shota Rustaveli National Science Foundation (Grant No 12/14).

REFERENCES

1. Vekua I. N. *Shell Theory: General Methods of onstruction*. Pitman Advanced Publishing Program, Boston-London-Melbourne (1985).

2. Vekua I. N. *On construction of approximate solutions of equations of shallow spherical shell.* Intern. J. Solid Structures, **5** (1969), 991-1003.

3. Meunargia T.V. On the application of the method of a small parameter in the theory of non-shallow I.N. Vekua's shells, *Proc. A. Razmadze Math. Inst.*, **141** (2006), 87-122.

4. Meunargia T.V. On one method of construction of geometrically and physically nonlinear theory of non-shallow shells. *Proc. A. Razmadze Math. Inst.*, **119** (1999), 133-154.

5. Gulua B.R. On construction of approximate solutions of equations of the non-linear and non-shallow cylindrical shells, *Bulletin of TICMI*, **13** (2009), 30-37.

Author's addresses:

B. Gulua
I. Vekua Institute of Applied Mathematics of
Iv. Javakhishvili Tbilisi State University
2, University St., Tbilisi 0186
Georgia

Sokhumi State University
9, Anna Politkovskaia St., Tbilisi 0186
Georgia
E-mail: bak.gulua@gmail.com