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TO ANALYTICAL AND NUMERICAL METHODS IN THE CONTINUUM MECHANICS

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Abstract. There are considered the method of approximate solution of boundary value problems for non homogeneous system of partial differential equations using analytical and optimal numerical methods as simultaneously processes.

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Let us consider the following problem:

$$L(\partial_x, \partial_y; x, y)u(x, y) = (A\partial_{xx} + 2B\partial_{xy} + C\partial_{yy} + a\partial_x + b\partial_y + c)u = f,$$

$$0 \le r < 1, \quad u|_{r=1} = g(x, y),$$

where $u = u(x, y)$ is unknown vector-function, for example ∂_x is partial derivative of first order, the other variable coefficients are known and coordinated with $u(x, y)$ It's evident that this problem is equivalent to another one:

$$\begin{aligned} &L(\partial_r, \partial_\phi; r, \phi)u = f, \quad u|_{r=1} = g(1, \phi), \\ &u(r, 0) = u(r, 2\pi), \ u(0, \phi) = const. \end{aligned} \tag{1}$$

Now we will use the finite-difference method (FDM) when $1 = mh$, $2\pi = n\tau$. As the circle mapped into rectangle and the boundary conditions (BC) will be defining exactly when $r = 1$. For simplicity and clarity we consider only five or nine point FD pattern. In this case the problem (1),with respect to exactness of remainder terms, are same to the following three-point operator expression:

$$\begin{aligned} &A_{1,i}U_{i+1} + A_{0,i}U_i + A_{-1,i}U_{i-1} = F_i, \quad i = m-1, m-2, ..., 1, \\ &A_{1,i} + A_{0,i} + A_{-1,i} = c, \\ &U_0 = U_0(0)(1, 1, ..., 1)^T, \quad U_i = (U_i(0), ..., U_i(n-1))^T. \end{aligned} \tag{2}$$

Here $A_{\alpha,i}$ ($\alpha = 0, \pm 1$) are Jacobi type cyclic matrices for which the first supper-row's last and the latter supper-row's first elements are in general nontrivial values.

Now we consider the following problem:

$$L^*(\partial_x, \partial_y)u + (L - L^*)u = f, \quad 0 \le r \le h, \ u|_{r=h} = g(h, \phi). \tag{3}$$

Here the linear operator $L^* = L^*(\partial_x, \partial_y)$ is a comparatively simple structure and in the certain sense similar to operator L. It's evident that if the mesh width is sufficient small the difference $L - L^* = \varepsilon M$, where M is fully defined similar to L or less order to him differential operator while $\varepsilon = \varepsilon(h)$ is a small parameter. By methodology of [1, pp.124-127] the scheme of find the approximate solution of (3) closed in the inversion of the operator L^* and the application εM to known vector-functions formulated by recursive precesses n times. The exactness of this methodology on the sufficient differentiable classes of functions is $O(\varepsilon^n)$. By this process the vector-function U_0 would be defined evidently.

It's possible to use for approximate solution of (3) the Green function and iteration methods. As far as L^* is differentialial operator, let us assume that for him is true the representation of corresponding solution by Green function method (for example, see [2-7]). Now the iteration process we write in the following form:

$$\begin{aligned} u^{[s]}(r,\phi) &= \int\limits_{\rho \le h} G_1(r,\varphi,\rho,\vartheta)[f(\rho,\vartheta) - (L - L^*)u^{s-1}(\rho,\vartheta)]d\rho d\vartheta \\ &- \int\limits_{\rho = h} G_2(r,\varphi,\vartheta)[g(h,\vartheta) \equiv u^{s-1}(h,\vartheta)]d\vartheta, \quad s = 0,1,2,.... \end{aligned} \tag{4}$$

where $G_1,\ G_2$-are the Green matrix-functions and $u^{[0]}(x,y)$ is an initial approximation.

As so h is arbitrary and may be sufficient small the process defining by (4) is convergent.

It's evident that for our aim is sufficient to use process (4) when $r = 0$ and thus we find approximately U_0.

Evidently that above-mentioned analytical-numerical scheme are true when we have circular sectors and segments, spherical, ellipsoidal, toroidal, cylindrical regions but not only Dirichlet type BC.

It's necessary to remark that the numerical schemes when the domains are rectangles or parallelepipeds, etc in polar, cylindrical or other curvilinear coordinates systems with classical BC were considered and investigated by many authors(for this is sufficient to cited typical monograph [8, pp.550-584]). Our considerations are essentially different from these investigations as well as we studied boundary value problems(BVP) with nonclassical BC.

As typical example we consider Dirichlet problem for Laplace equation in the circle

$$\Delta u(x,y) = 0,\ 0 \le x^2 + y^2 < R^2,\ u|_{r=R} = g(x,y). \qquad (A)$$

There are well-known that the solution of this this problem is representing in two equivalent forms: by Poisson Integral (PI) and Trigonometrical Series(TS).

Let us assume that the aim of an user is tabulation of function $u(x,y)$ by PI. If the calculate of integrals are difficult or impossible for tabulation are using quadrature formulas dividing $(0, 2\pi)$ into n part. The order of arithmetical operations (AOs) for find the approximate solution is $O(n)$ *Horner* (H). As the number of net points are $\approx n^2$, for tabulation of unknown function is necessary $O(n^3)$ H AOs (here we neglected the additional AOs connected with calculation of $\cos(\varphi - \vartheta_k)$ functions).

In case of applied TS for find its coefficients by quadrature formulas and tabulation of unknown function it's necessary $O(n^4 \log_2 n)$ order of AOs if we using "The Fast Fourier Transformation" too.

Below we consider the problem of approximate solution of (A) by above described methodology, which also has illustrating character.

Instead of BVP (A) if we introduced polar coordinates, we have:

$$r\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{\partial^2 u}{\partial \varphi^2}=0,\ u(0,\varphi)=u(0,0)=\frac{u_0'}{2}, \qquad (B)$$

$$u(r,0)=u(r,2\pi),\ u(R,\varphi)=g(\varphi),\ u_0'=\frac{1}{\pi}\int\limits_0^{2\pi} g(\vartheta)d\vartheta.$$

Now for clarity we use FDM of second order accuracy for BVP (B). Let us prove that for tabulation of the approximate solution it's sufficient $O[\max(h,\tau)^{-2}]$ H AOs where $hm=R,\ n\tau=2\pi$.

From (B) follows:

$$((r_ih^{-1})^2\delta_r^2+r_ih^{-1}\delta_r+\tau^{-2}\delta_\varphi^2)u_{i,j}=O(h^2+\tau^2), \qquad (5)$$

$$i=1,2,...,m-1,\ \ j=1,2,...,n.$$

where $\delta,\ \delta^2$ are the first and second order symmetric difference operators. If we denote $U_i=(u_{i,1},u_{i,2},...,u_{i,m})^T$ then (5), without remainder members and by using BC, has the following form:

$$E_{i-1}U_{i-1}-A_iU_i+E_{i+1}U_{i+1}=0,\ \ i=1,2,...,m-1, \qquad (6)$$

where

$$A_i=\{a_{kj}\}_{n\times n},\qquad a_{kk}=2(1+r_i^2\tau^2/h^2),$$

$$a_{k-1,k}=a_{k+1,k}=a_{1n}=a_{n1}=-1,\ \ a_{kj}=0,\ j\neq k-1,k,k+1,$$

$$E_{i-1}=r_i\tau^2/h(r_i/h-0,5)E,$$

$$E_{i+1}=r_i\tau^2/h(r_i/h+0,5)E,\ E=\{1,1,...,1\},$$

$$U_0=0,5u_0'(1,1,...,1)^T,\ \ U_m=(g(\tau),g(2\tau),...,g(n\tau))^T.$$

We see that the matrix corresponding to system (5) is incomposable and (by theorem of O. Tausski) nondegenerate too as the criterion of diagonal elements domination property are true for first and latter vectorial equations when $i=1,m-1$ in (6). As we see the matrices A_i are cyclical types. In this case, same algorithm as of three diagonal Jacobi matrices, for construction of the factorization Gauss type scheme is also true and directly applicable (see, for example, [9, pp.18,19]) and thus the AOs $O(m)$H. Let us calculate now the order of AOs for tabulation of unknown vector $U=(U_1,U_2,...,U_{m-1})^T$. For this aim let us use the Gauss factorization scheme for tree-point operator equation (for example, see [8, pp.103-120]). These processes are required the inversion of A_i type cyclic matrices. On the next step we give new cyclic matrix. The formulation of such matrices are required $O(n)$ H AOs. By the process of multiplications on diagonal type matrix E_i the order of AOs is not change. As the number of vector equations is m the whole AOs (multiplications and divisions) for tabulation of approximate solution of (B) have $O[\max(h,\tau)^{-2}]$ order.

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